LECTURE 1

Sequences

A sequence is an ordered set of numbers.

Each number in the set is called a term of the sequence.

An infinite sequence is a sequence with an infinite number of terms.

Sequence notation: $a_1, a_2, , , , .a_n$ or $\{a_n\}$.

The first term of a sequence is denoted a_1. The second term a_2. The term in the n-th position, called the n th term, is denoted by a_n.

Examples:

1) The sequence 1,4,9,16,25,…can be specified by the formula $a_n = n^2$.

2) The sequence 2,4,6,8,10,… can be specified by the formula $a_n = 2n$.

Bounded sequences

A sequence $\{a_n\}$ is said to be bounded above if there is a number M such that $a_n \leq M$ for all n.

In short, in terms of mathematics it can be written as follows:

$\{a_n\}$is bounded above $\Leftrightarrow \exists M \ \forall n : a_n \leq M$.

A sequence $\{a_n\}$ is said to be bounded below if there is a number m such that $a_n \geq m$ for all n.

Similarly, it can be written as follows:

$\{a_n\}$ is bounded below $\Leftrightarrow \exists m \ \forall n : a_n \geq m$.

Examples:

1) $a_n = -n$ is bounded above.

2) $a_n = n^2$ is bounded below.

A sequence both above and below is called bounded sequence. In other words, a sequence $\{a_n\}$ is said to be bounded if there exist numbers M and m such that $m \leq a_n \leq M$ for all n. In short, it can be written as $\{a_n\}$ is bounded $\Leftrightarrow \exists M, m \quad \forall n : m \leq a_n \leq M$.

Examples:

1) Consider the sequence whose terms are defined by $a_n = \frac{1}{n}$. It is that $0 < \frac{1}{n} \leq 1$. Therefore, the given sequence a_n is bounded.

2) A sequence is defined by $a_n = (-1)^n$.

We see that $a_1 = -1, \ a_2 = 1, a_3 = -1, a_4 = 1,$ and so on. Since this sequence assumes only two values -1 and 1 we can write $-1 \leq a_n \leq 1$. Therefore, the given sequence a_n is bounded.

Now we introduce another equivalent definition of bounded sequence.

A sequence $\{a_n\}$ is said to be bounded if there is a positive number $C > 0$ such that $|a_n| \le C$ for all n.

$\{a_n\}$ is bounded $\Leftrightarrow \exists C > 0, \forall n : \quad |a_n| \le C$.

A sequence is said to be unbounded if it is unbounded either from above or below. In other words, a sequence $\{a_n\}$ is said to be unbounded if given any positive number $C > 0$ there is a number $n \in N$ such that $|a_n| > C$.

Examples of unbounded sequences are the sequences $a_n = n^2,\ a_n = 2n,\ a_n = -n,\quad a_n = (-1)^n \cdot n,\quad a_n = \sqrt{n}\,.a$

Limit of a sequence

Definition. A number a is said to be the limit of the sequence $\{a_n\}$, $a_1, a_2, \ldots, a_n, \ldots$ as $n \to \infty$ if for all sufficiently large values of n the corresponding values a_n of the sequence $\{a_n\}$ become arbitrary close to the number a.

If a is the limit of the sequence $\{a_n\}$ as $n \to \infty$ we write

$\lim_{n\to\infty} a_n = a$ or $a_n \underset{n\to\infty}{\to} a$

Now we give a mathematically more rigorous definition of the limit.

Definition. A number a is said to be the limit of the sequence $\{a_n\}$, as $n \to \infty$ if for any positive number ε (however small) there is a positive number N such that the inequality $|a_n - a| < \varepsilon$ holds for all $n > N.$ thus, we have

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \forall \varepsilon > 0 \ \exists N > 0 \ \ \forall n > N : |a_n - a| < \varepsilon$$

To understand the deep meaning of the concept of limit, it is advisable to rephrase the definition in terms of "neighborhoods." Recall that the neighborhood of a point is any interval containing this point.

It is clear that

$$|x_n - a| < \varepsilon \Leftrightarrow -\varepsilon < x_n - a < \varepsilon \ \Leftrightarrow$$
$$a - \varepsilon < x_n < a + \varepsilon \Leftrightarrow$$
$$x_n \in (a - \varepsilon, a + \varepsilon) = U(a),$$

where $U(a)$ is a neighborhood of a point a.

We are now ready to define the concept of limit in terms of neighborhoods.

Definition. A number a is said to be the limit of the sequence $\{x_n\}$ if for any neighborhood $U(a)$ of the point a there exist a positive number N such that $x_n \in U(a)$ for all $n > N$.

$$\lim_{n\to\infty} x_n = a \Leftrightarrow \forall U(a) \ \exists N \ \forall n > N : x_n \in U(a)$$

In other words this means that.

A number a is said to be the limit of the sequence $\{x_n\}$ if for any neighborhood $U(a)$ of the point a outside it remain only a finite number of terms of this sequence.

Example. The limit of the sequence

$$\frac{1}{2}, \frac{5}{7}, \frac{7}{8}, \ldots, \frac{2n+1}{n+5}, \ldots$$

exists and is equal to 2. Indeed, for the absolute value of the difference between $\frac{2n+1}{n+5}$ and 2, which is equal to

$$\left|\frac{2n+1}{n+5}-2\right|=\frac{9}{n+5}$$

to be less than a given positive number ε, it is sufficient that the inequality $n+5>\frac{9}{\varepsilon}$ should be fulfilled. This is equivalent to the condition $n>\frac{9}{\varepsilon}-5$, and hence, given ε, there exists $N=\frac{9}{\varepsilon}-5$ such that for all $n>N$ the absolute value of the above difference is less then ε, which means that $\lim\limits_{n\to\infty}\frac{2n+1}{n+5}=2$.

Definition. A sequence that has a finite limit is called convergent.

If otherwise, the sequence is said to be divergent.

Definition. A sequence $\{a_n\}$ is said to be infinitely large if, for any positive number ε (however large) there is a positive number N such that the inequality $|a_n|>\varepsilon$ holds for all $n>N$.

Hence, $\lim\limits_{n\to\infty}a_n=\infty\Leftrightarrow\forall\varepsilon>0\ \ \exists N>0\ \ \forall n>N:|a_n|>\varepsilon$

Similarly, the definition of the limit of the sequence is paraphrased for the case when this limit is infinite with a definite sign.

$\lim\limits_{n\to\infty}a_n=+\infty\Leftrightarrow\forall\varepsilon>0\ \ \exists N>0\ \ \forall n>N:\ \ a_n>\varepsilon$

$\lim\limits_{n\to\infty}a_n=-\infty\Leftrightarrow\forall\varepsilon>0\ \ \exists N>0\ \ \forall n>N:\ \ a_n<-\varepsilon$

Obviously, if $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$

then $\lim_{n\to\infty} a_n = \infty$

Examples of infinitely large sequences are

$a_n = n,\ a_n = -n,\ a_n = n^2,\ a_n = (-1)^n \cdot n,\ a_n = 2n+1,\ a_n = (-2n)^{2n}$

$a_n = (-(2n-1))^{2n-1};\ a_n = (-n)^n$.

Moreover, it is clear that

$\lim_{n\to\infty} n = +\infty,$ $\lim_{n\to\infty}(-2n)^{2n} = +\infty$

$\lim_{n\to\infty}(-n) = -\infty,$ $\lim_{n\to\infty}(-(2n-1)^{2n-1} = -\infty$

$\lim_{n\to\infty} n^2 = +\infty,$ $\lim_{n\to\infty}(-n)^n = \infty$

$\lim_{n\to\infty}(-1)^n \cdot n = \infty$

Infinitesimals

Definition. A sequence α_n tending to zero as $n \to \infty$ is called an infinitesimal.

$\{\alpha_n\}$ is an infinitesimal $\Leftrightarrow \lim_{n\to\infty} \alpha_n = 0$. Examples of infinitesimals are $a_n = \frac{1}{n}$,

$a_n = \frac{(-1)^n}{n},\ a_n = \frac{1}{n^2},\ a_n = \frac{1}{\sqrt{n}},\ a_n = \frac{1}{n!},\ a_n = \left(\frac{1}{2}\right)^n$.

Now we introduce a relation of equality between sequences and operations on them.

Definition. $\{x_n\}=\{y_n\} \Leftrightarrow x_n = y_n \quad \forall n$

$$\{x_n\} \pm \{y_n\} = \{x_n \pm y_n\}$$

$$\{x_n\} \cdot \{y_n\} = \{x_n \cdot y_n\}$$

$$\frac{\{x_n\}}{\{y_n\}} = \left\{\frac{x_n}{y_n}\right\}, \quad (y_n \neq 0,\ \forall n)$$

$$\lambda\{x_n\} = \{\lambda x_n\}$$

Theorem. A sum, a difference and a product of two infinitesimals is again an infinitesimal.

In other words, if $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimals then $\{\alpha_n + \beta_n\}, \{\alpha_n - \beta_n\}$ and $\{\alpha_n \cdot \beta_n\}$ are also infinitesimals.

Theorem. The product of a bounded sequence by an infinitesimal is an infinitesimal. $\{x_n\}$ is bounded and $\{\alpha_n\}$ is an infinitesimal then $\{\alpha_n x_n\}$ is a infinitesimal.

Example. Let us consider the sequence $\left\{\frac{\sin n}{n}\right\}$. Which can be written as

$$\frac{\sin n}{n} = \sin n \cdot \frac{1}{n}$$

It is clear that $x_n = \sin n$ is bounded since $-1 \leq \sin n \leq 1$, and $\alpha_n = \frac{1}{n}$ is an infinitesimal. Therefore the product $\alpha_n x_n = \frac{\sin n}{n}$ is an infinitesimal.

The theorems below are important for applications.

Direct theorem. If a sequence has a finite limit it is representable as a sum of a constant, equal to that limit, and an infinitesimal.

$$\lim_{n\to\infty} x_n = a \Rightarrow x_n = a + \alpha_n \text{ where } \lim_{n\to\infty} \alpha_n = 0$$

Reciprocal theorem. If a sequence is representable as a sum of a constant and an infinitesimal the constant summand is the limit of the sequence.

$x_n = a + \alpha_n$, where $\lim_{n\to\infty} \alpha_n = 0 \Rightarrow \lim_{n\to\infty} x_n = a$

Properties of limits

1. If $x_n = C$, C - constant $\forall n$ then $\lim_{n\to\infty} x_n = C$. This means that the limit of a constant equals the constant.

2. If a sequence has a limit then this limit is unique, i.e., there no other limits of the sequences.

3. If a sequence has a limit then it is bounded.

Proof. Let $\{x_n\}$ be convergent and $\lim_{n\to\infty} x_n = a$. Take $\varepsilon = 1$. According to definition there exist a number N such that the inequality $|x_n - a| < 1$ holds for all $n > N$. Let $d = \max\{1, |x_1 - a|, \ldots, |x_n - a|\}$. Then the inequality $|x_n - a| \le d$ holds for all n.

The last inequality can be written as

$$-d \le x_n - a \le d$$

or $a - d \le x_n \le a + d$

denoting $m = a - d$ and $M = a + d$ we get

$m \le x_n \le M$ for all n.

This means that $\{x_n\}$ is bounded.

4. The limit of a sum of sequences is equal to the sum of the limits of these sequences.

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} y_n = b$

According to the theorem we can write

$x_n = a + \alpha_n$ and $y_n = b + \beta_n$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimals.

It follows that

$x_n + y_n = a + b + \alpha_n + \beta_n$ or

$x_n + y_n = a + b + \gamma_n$ where $\gamma_n = \alpha_n + \beta_n$ and $\lim_{n\to\infty} \gamma_n = 0$

This means that $\lim_{n\to\infty}(x_n + y_n) = a + b = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

5. The limit of a difference of sequences is equal o the difference of the limits of these sequences:

$$\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$$

Proof. Let $\{x_n\}$ and $\{y_n\}$ be convergent and $\lim_{n\to\infty} x_n = a \quad \lim_{n\to\infty} y_n = b$

Then we have $x_n = a + \alpha_n$ and $y_n = b + \beta_n$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimals.

It follows that

$x_n - y_n = a - b + \alpha_n - \beta_n$ or

$x_n - y_n = a - b + \gamma_n$

where $\gamma_n = \alpha_n - \beta_n$ and $\lim_{n\to\infty} \gamma_n = 0$

Hence, $\lim_{n\to\infty}(x_n - y_n) = a - b = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$

6. The limit of a product of sequences is equal to the product of the limits of these sequences.

$\lim_{n\to\infty} x_n y_n = a - b = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$

Proof. Let $\{x_n\}$ and $\{y_n\}$ be convergent and $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$

Then we can write

$x_n = a + \alpha_n$ and $y_n = b + \beta_n$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimals.

It follows that

$x_n y_n = (a + \alpha_n) \cdot (b + \beta_n) = ab + (\alpha_n b + \beta_n a + \alpha_n \beta_n) = ab + \gamma_n$

where $\gamma_n = \alpha_n b + \beta_n a + \alpha_n \beta_n$

According to the properties of infinitesimals it is clear that $\lim_{n\to\infty} \gamma_n = 0$. Hence

$$\lim_{n\to\infty} x_n y_n = a \cdot b = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

7. A constant factor may be taken outside the sign of the limit.

$\lim_{n\to\infty} Cx_n = C \cdot \lim_{n\to\infty} x_n$ where C is constant.

This property follows from properties 1 and 6 .

8. The limit of a ratio of sequences is equal to the ratio of the limits of these sequences.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} \quad (y_n \neq 0 \text{ and } \lim_{n\to\infty} y_n \neq 0)$$

9. If $x_n \le y_n \le z_n$ for all n and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$ then $\lim_{n\to\infty} y_n = a$

Proof. We have

$$\lim_{n\to\infty} x_n = a \Rightarrow \forall \varepsilon > 0 \ \exists N_1 \ \ \forall n > N_1 : |x_n - a| < \varepsilon$$

$$\lim_{n\to\infty} z_n = a \Rightarrow \forall \varepsilon > 0 \ \exists N_2 \ \ \forall n > N_2 : |z_n - a| < \varepsilon$$

Now putting $N = \max\{N_1, N_2\}$ we get

$|x_n - a| < \varepsilon$ and $|z_n - a| < \varepsilon$ for all $n > N$.

It follows that

$$a - \varepsilon < x_n \le y_n \le z_n < a + \varepsilon$$

or $a-\varepsilon < y_n < a+\varepsilon$

or $|y_n - a| < \varepsilon$ for all $n > N$

Hence we get

$\forall \varepsilon > 0 \;\; \exists N \;\; \forall n > N: \quad |y_n - a| \le \varepsilon$.

By definition, this means $\lim\limits_{n\to\infty} y_n = a$. and we get what we had to prove.

10. Theorem.

$$\lim_{n\to\infty} \frac{a_0 n^p + a_1 n^{p-1} + \ldots + a_k}{b_0 n^q + b_1 n^{q-1} + \ldots + b_s} = \begin{cases} \dfrac{a_0}{b_0} \;\; for \quad p = q, \\ \infty, \; for \quad p > q, \\ 0, \; for \quad p < q \end{cases}$$

Monotone sequences

Definition. A sequence $\{x_n\}$ is said to be increasing if the inequality $x_n \le x_{n+1}$ holds for all n.

Examples of increasing sequences are

$x_n = n, \quad x_n = 2^n, \quad x_n = \sqrt{n}, \quad x_n = n^2$

Definition. A sequence $\{x_n\}$ is said to be decreasing if the inequality $x_n \geq x_{n+1}$ holds for all n. Examples of decreasing sequences are

$x_n = \frac{1}{n}, \quad x_n = -n, \quad x_n = \frac{1}{3^n}, \quad x_n = \frac{1}{\sqrt{n}}$

Definition. Increasing and decreasing sequences are called monotone.

Note. Not every sequence is monotone. For example, the sequences $\{(-1)^n\}$ and $\left\{(-1)^n \frac{1}{n}\right\}$ are not monotone.

Theorem. A monotone and bounded sequence is convergent. This theorem can be rephrased as follows.

Theorem. An increasing and bounded above sequence has a finite limit.

A decreasing and bounded below sequence has a finite limit.

Number e

Consider the sequence $x_n = \left(1 + \frac{1}{n}\right)^n, \; n = 1,2,,...$

We shall prove that $\{x_n\}$ is convergent.

By Newton's binomial formula, we have

$$x_n = \left(1+\frac{1}{n}\right)^n = 1+\frac{n}{1!}\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\ldots+\frac{n(n-1)\ldots(n-n+1)}{n!}\cdot\frac{1}{n^n}=$$
$$1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right)$$

Replacing n by $n+1$ we obtain an analogous expression for x_{n+1}.

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1+\frac{n+1}{1!}\cdot\frac{1}{n+1}+\frac{(n+1)\cdot n}{2!}\cdot\frac{1}{(n+1)^2}+\frac{(n+1)\cdot n\cdot(n-1)}{3!}\cdot\frac{1}{(n+1)^3}+\ldots$$
$$\ldots+\frac{(n+1)\cdot n(n-1)\ldots(n+1-n)}{(n+1)!}\cdot\frac{1}{(n+1)^{n+1}} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\ldots$$
$$\ldots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n}{n+1}\right)$$

Now, comparing these expressions for x_n and x_{n+1} we conclude that the first two terms in both sums coincide and that every subsequent term in x_n is less than the corresponding term in x_{n+1} since

$$1-\frac{k}{n}<1-\frac{k}{n+1},\ k=1,2,\ldots,n-1$$

Therefore $x_n < x_{n+1}$ for any n, that is x_n is an increasing sequence.

Let us show that it is bounded. It is clear that

$$x_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}<$$
$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}+\frac{1}{2^n}+\ldots<1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots\right)$$

The sum of the progression in the parentheses being equal to 2 we obtain $x_n < 3$.

Thus, the increasing sequence $\{x_n\}$ is bounded above, hence according to the above theorem, it possesses a finite limit.

Thus, we have proved the following theorem.

Theorem. The sequence $x_n = \left(1+\frac{1}{n}\right)^n$ possesses a limit as $n \to \infty$.

This limit plays a very important role in mathematics and is traditionally denoted by *e*.

Definition. The number *e* is the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

The number e is irrational and therefore cannot be expressed by a finite decimal. Its approximate value is

$$e = 2{,}71828182845... \approx 2{,}718$$

Lecture 3

Limit of a function

Definition. *A* number *A* is said to be the limit of the function $f(x)$ as $x \to a$ if for all the values of *x* lying sufficiently close to a the corresponding values of the function $f(x)$ are arbitrary close to the number A. It can be written as

$$\lim_{x\to a} f(x) = A \text{ or } f(x) \underset{x\to a}{\to} A$$

It should be noted that this definition does not require that the function should be defined at the point a itself. It is only necessary that the function should be defined in an arbitrary neighborhood of this point. Moreover, the phrase "arbitrary close" means "as close as you want". Now let us introduce the following

Definition. A number A is said to be the limit of the function $f(x)$ as $x \to a$ if for given any positive number ε (however small), there is a positive number δ such that for all *x* different from *a* and satisfying the inequality $0 < |x - a| < \delta$ there holds the inequality

$$|f(x) - A| < \varepsilon$$

In short

$$\lim_{x \to a} f(x) = A \Leftrightarrow \ \forall \varepsilon > 0 \quad \exists \delta > 0 \ \ \forall x : \ 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon$$

This definition is sufficiently rigorous because the imprecise terms "arbitrary close" and "sufficiently close" in the previous definition have been replaced by a precise statement using inequalities.

Example: Let $f(x) = 5x - 3$ and prove that $\lim_{x \to 1} f(x) = 2$

To this end, choose a positive number ε. For the inequality

$$|f(x) - 2| < \varepsilon \ \text{ or } |(5x - 3) - 2| < \varepsilon$$

or the equivalent inequality

$$5 \cdot |x - 1| < \varepsilon$$

to be fulfilled it is sufficient that the condition

$$|x-1|<\frac{\varepsilon}{5}$$

should hold.

Hence, for all *x* differing from 1 by less than $\frac{\varepsilon}{5}$ the function $f(x)$ differs from 2 by less than ε where ε is an arbitrary positive number.

In this example we can put $\delta=\frac{\varepsilon}{5}$. Therefore

$$\forall\varepsilon>0 \quad \exists\delta=\frac{\varepsilon}{5}>0 \quad \forall x:\ 0<|x-1|<\delta \Rightarrow |f(x)-2|<\varepsilon.$$

According to definition this means that $\lim\limits_{x\to 1} f(x)=2$. It should be noted that here it is not necessary to stipulate the condition $x\neq 1$ since inequality $|x-1|<\frac{\varepsilon}{5}$ is also fulfilled for $x=1$.

In this example the limit of the function $f(x)$ as x approaches 1 is equal to the value of the function at that point.

$$\lim_{x\to 1} f(x)=2=f(1).$$

Note. If the limit of a function $f(x)$ as x approaches a exists, this limit may not be equal to $f(a)$. In fact, $f(a)$ may not even be defined.

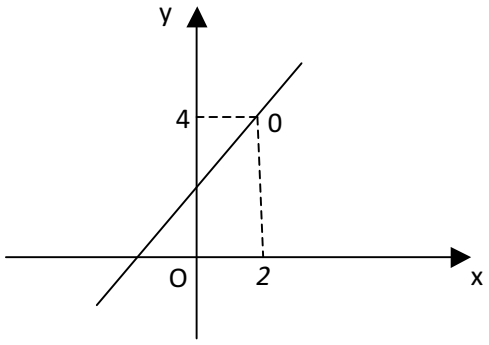
Example. Let us consider the function $f(x)=\frac{x^2-4}{x-2}$ which is defined for all *x* except 2 ($x\neq 2$).

It is clear that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4 \neq f(2)$$

In this example the limit exist and is not equal to the value of the function at the given point 2. Moreover, $f(2)$ is not defined.

Figure shows the graph of $f(x) = \frac{x^2-4}{x-2},\ x \neq 2,$ near $x = 2$. As *x* approaches 2 from the left or right, the corresponding values of $f(x)$ becomes closer to 4.



Properties of limits

These are a number of facts that simplify the computation of limits.

1. If $f(x) = C$ (*C*-constant) then $\lim_{x\to a} f(x) = C$.

2. If $f(x) = x$ then for every number *a*, $\lim_{x\to a} x = a$

3. If $g(x) \le f(x) \le h(x)$ and there exist $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = A$ then $\lim_{x\to a} f(x) = A$

4. If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

(a) $\lim_{x\to a}(f+g)(x) = \lim_{x\to a}\left[f(x)+g(x)\right] = A+B$

(b) $\lim_{x\to a}(f-g)(x) = \lim_{x\to a}\left[f(x)-g(x)\right] = A-B$

(c) $\lim_{x\to a}(fg)(x) = \lim_{x\to a}\left[f(x)g(x)\right] = A\cdot B$

(a) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{A}{B}$ (for $B \neq 0$)

(e) $\lim_{x\to a}\sqrt{f(x)} = \sqrt{A}$ ($for\ f(x) \geq 0$ for all x near a)

5. If $f(x)$ is a polynominal function, that is

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$,

Then $\lim_{x\to a} f(x) = f(a)$ for any real number *a*.

6. Let $f(x)$ be a rational function, that is

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m}$$

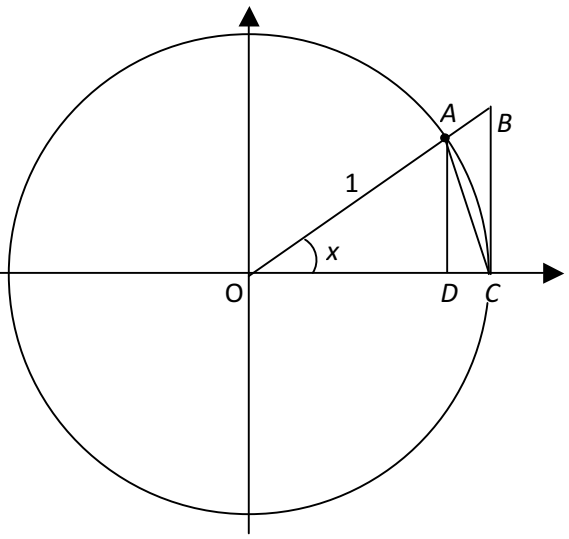
and let a be a real number such that $f(a)$ is defined. Then $\lim_{x\to a} f(x) = f(a)$

7. $$\lim_{x\to\infty}\frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m} = \begin{cases} \frac{a_0}{b_0} & for\ n = m \\ \infty & for\ n > m \\ 0 & for\ n < m \end{cases}$$

Remarkable limits

Theorem. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Proof. Consider a circle of unit radius suppose that the control angle x expressed in radians and $0<x<\dfrac{\pi}{2}$. From the figure it is clear that



$$Area_{\Delta ACO} < Area_{sectorACO} < Area_{\Delta BCO}$$

We have

$$Area_{\Delta ACO}=\frac{1}{2}OC\cdot AD=\frac{1}{2}\cdot 1\cdot\sin x=\frac{1}{2}\sin x$$

$$Area_{sectorACO}=\frac{1}{2}OC^2\cdot\overset{\smile}{AC}=\frac{1}{2}\cdot 1\cdot x=\frac{1}{2}x$$

$$Area_{\Delta BCO}=\frac{1}{2}\cdot OC\cdot BC=\frac{1}{2}\cdot 1\cdot\tan x=\frac{1}{2}\tan x$$

It follows that $\dfrac{1}{2}\sin x<\dfrac{1}{2}x<\dfrac{1}{2}\tan x$

or $\sin x<x<\tan x$

Dividing all the terms of these inequalities by $\sin x$ we get

$$1<\frac{x}{\sin x}<\frac{1}{\cos x}$$

which can be written as

$$\cos x<\frac{\sin x}{x}<1$$

Since $\lim_{x\to 0}\cos x = 1$ and $\lim_{x\to 0}1 = 1$ passing to the limit as $x \to 0$ we get, in accordance with properties of limits, the desired result: $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Many other limits can be computed with the aid of this limit.

Examples

1) $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} == 1\cdot 1 = 1$

2) $\lim_{x\to 0}\frac{\sin kx}{x} = \lim_{x\to 0}\frac{k\cdot\sin kx}{kx} = k\lim_{x\to 0}\frac{\sin kx}{kx} = k\cdot 1 = k$

3) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2\cdot 1 = 2$

4) $\lim_{x\to 0}\frac{\arctan x}{x} = \lim_{x\to 0}\frac{\arctan x}{\tan(\arctan x)}$

Putting $\arctan x = t,$ it is clear that $t \to 0$ as $x \to 0$ we obtain

$$\lim_{x\to 0}\frac{\arctan x}{x} = \lim_{t\to 0}\frac{t}{\tan t} = \lim_{t\to 0}\frac{1}{\frac{\tan t}{t}} = \frac{1}{1} = 1$$

5) $\lim_{x\to 0}\frac{\arcsin x}{x} = 1$

$\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{x\to 0}\frac{\arcsin x}{\sin(\arcsin x)} = \lim_{y\to 0}\frac{y}{\sin y} = 1$

The limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ playing an important role in mathematics is called the first remarkable limit.

Theorem. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Many other limits can be computed with the aid of this limit. Examples:

1) $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^x = e^k$

2) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{kx} = e^k$

3) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

4) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = \lim\limits_{x\to 0}\frac{1}{x}\ln(1+x) = \lim\limits_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \ln e = 1$

5) $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a$

Indeed, putting $a^x - 1 = t \to 0$ as $x \to 0$. We have $a^x = 1+t$ hence $x\ln a = \ln(1+t)$ therefore $x = \frac{1}{\ln a}\cdot\ln(1+t)$.

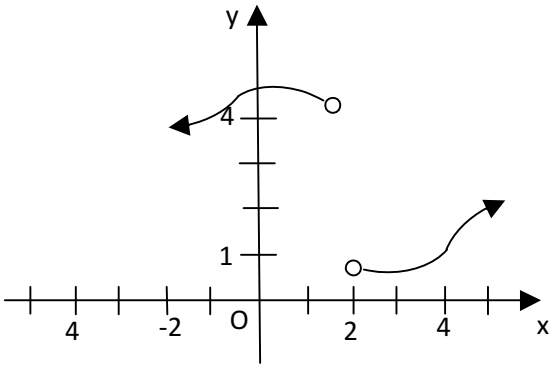
Making substitution we get

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{t\to 0}\frac{t}{\frac{1}{\ln a}\cdot\ln(1+t)} = \ln a\cdot\lim_{t\to 0}\frac{t}{\ln(1+t)} = \ln a$$

The limit $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ has a wide application and is called the second remarkable limit.

One – sided limits

The function whose graph is shown below is defined for all values of *x* except *x*=2.



As x approaches 2 from the right, the graph shows that the corresponding values of $f(x)$ get very close to 1. Consequently,

$$\lim_{\substack{x\to 2\\ x>2}} f(x)=1 \quad \text{or} \quad \lim_{x\to 2+0} f(x)=1$$

The notation $x\to 2+0$ indicates that only the values of *x* with $x>2$ are considered. Similarly, as *x* approaches 2 from the left, the graph shows that the corresponding values of $f(x)$ get very close to 4. Consequently,

$$\lim_{\substack{x\to 2\\ x<2}} f(x)=4 \quad \text{or} \quad \lim_{x\to 2-0} f(x)=4$$

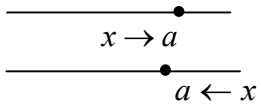
Here, the notation $x\to 2-0$ indicates that only the values of x with $x<2$ are considered.

These “one-sided” limits are a bit different than the “two sided” limits discussed in previous sections. When *x* approaches 2 from the left and from the

right, the corresponding values of $f(x)$ do not approach a single number, so the limit as $x \to 2,$ i.e. $\lim_{x\to 2} f(x)$ does not exist.

Let us introduce the following notations:

$x \to a-0 \Leftrightarrow x \to a$ and $x < a$
$x \to a+0 \Leftrightarrow x \to a$ and $x > a$
$f(a-0) = \lim\limits_{x\to a-0} f(x) = \lim\limits_{\substack{x\to a \\ x<a}} f(x)$
$f(a+0) = \lim\limits_{x\to a+0} f(x) = \lim\limits_{\substack{x\to a \\ x>a}} f(x)$



Definition. The numbers $f(a-0)$ and $f(a+0)$ are called respectively the left-hand limit and the right-hand limit provided that the corresponding limits exist.

The computation of one-sided limits is greatly facilitated by the following fact.

All the results about limits in previous sections, such as the properties of limits, limits of polynomial functions remain valid if "$x \to a$" is replaced by either "$x \to a-0$" or "$x \to a+0$".

Now we state the following necessary and sufficient condition for existence of limit.

Theorem.

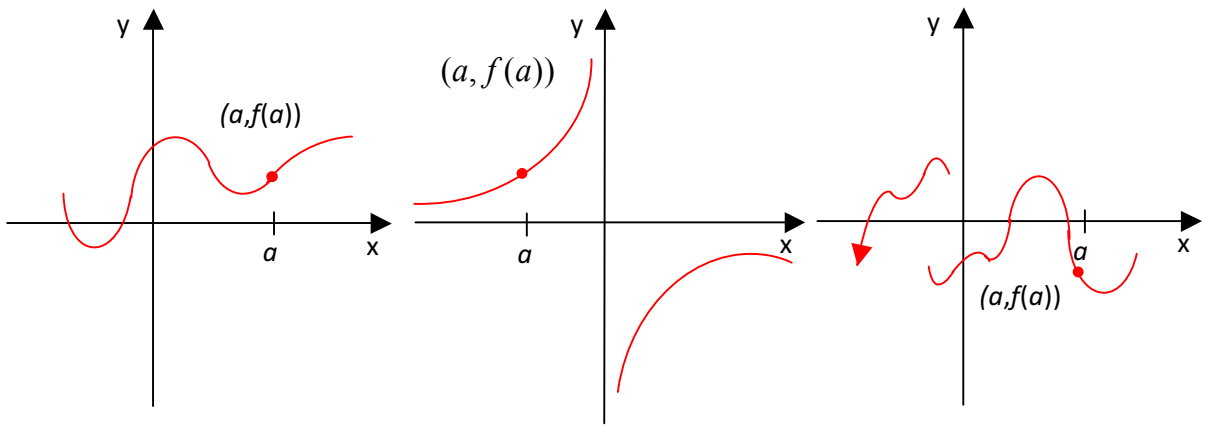
$$\exists \lim_{x \to a} f(x) = A \Leftrightarrow \exists f(a-0) \quad \exists f(a+0) \text{ and } \quad f(a-0) = f(a+0) = A$$

It means that if a function $f(x)$ possesses a limit as $x \to a$ (i.e. x approaches a in an arbitrary way) the left-hand and the right-hand limits also exist and coincide with the limit of the function. Conversely, if one-sided limits $f(a-0)$ and $f(a+0)$ exist and coincide $f(a-0) = f(a+0) = A$ then the function $f(x)$ has the same limit *A* as *x* the argument *x* approaches a arbitrary.

Lecture 4

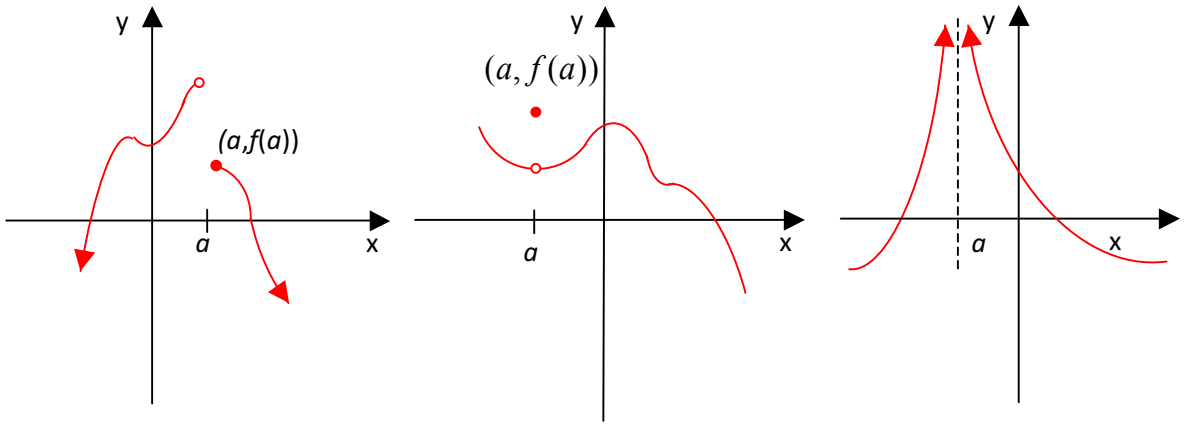
Continuity of a function

Let *a* be a real number in the domain of a function *f*. Informally, the function *f* is continuous at $x = a$ if you can draw the graph of *f* at and near the point $(a, f(a))$ without lifting your pencil from the paper. For example, each of the following graphs is the graph of a function that continuous at $x = a$.



Thus, a function is continuous at $x = a$ if its graph around the point $(a, f(a))$ is connected and unbroken.

On the other hand, none of the functions whose graphs are shown below is continuous at $x = a$.



Each of the functions is discontinuous, that is, not continuous at $x = a$ because you cannot draw the graph of *f* at and near the point $(a, f(a))$ without lifting your pencil from the paper. In other words, a function *f* is not continuous at $x = a$ if its graph has a jump, break, gap, or hole when $x = a$.

Now we state the following formal definition of continuity.

Definition. A function $y = f(x)$is said to be continuous at a point *a* if it is defined in a neighbourhood of the of the point *a* and $\lim_{x \to a} f(x) = f(a)$.

This definition we can restate as follows.

Definition. A function $y = f(x)$ is said to be continuous at a point *a* under the following conditions:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$.

Here is another equivalent definition of continuity:

Definition. A function $y = f(x)$ is said to be continuous at a point *a* if for given any $\varepsilon > 0$, there is $\delta > 0$ such that for all the values of x satisfying the condition $|x - a| < \delta$ the inequality $|f(x) - f(a)| < \varepsilon$ is fulfilled.

Properties of continuous functions

Theorem. If the functions *f* and *g* are continuous at $x = a$, then each of the following functions

1. The sum function *f +g*
2. The difference function $f - g$

3. The product function *f g*

4. The quotient function $\frac{f}{g},\ g(a) \neq 0$

is also continuous at $x = a$.

Continuity of special functions

Theorem 1. Every polynomial function, that is

$$f(x) = a_0 x^n + a_n x^{n-1} + ... + a_n,$$

is continuous at every real number.

Theorem 2. Every rational function, that is

$$f(x) = \frac{a_0 x^n + a_1 x^{n-1} + ... + a_n}{b_0 x^m + b_1 x^{m-1} + ... + b_m}$$

is continuous at every real number in its domain.

Theorem 3. Every exponential function $y = a^x$ is continuous at every real number.

Theorem 4. Every logarithmic function $y = \log_a x$ is continuous at every positive real number.

Theorem 5. The trigonometric function $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every real number.

Theorem 6. The trigonometric function $f(x) = \tan x$ and $g(x) = \cot x$ are continuous at every real number in its domain.

Definition. A function f is said to be continuous on an interval if it is continuous at every point of that interval.

It can be shown that all the basic elementary functions are continuous on the intervals where they are defined.

Points of discontinuity of a function

Definition. If, for a function $y = f(x)$, the condition of continuity, at a point *a*, is violated, *a* is called a point of discontinuity of the function.

For better understanding, we introduce the following more precise definition.

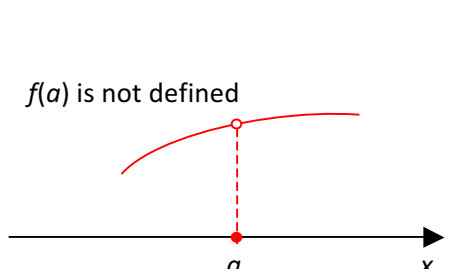
Definition. If, for a function $y = f(x)$, at least one of the conditions of continuity, at a point *a*,

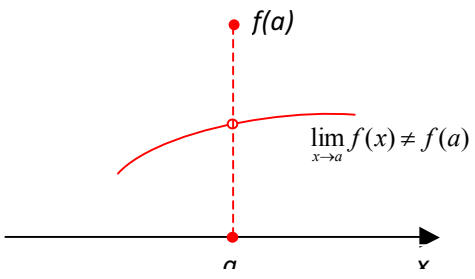
1. $f(a)$ is defined
2. $\lim\limits_{x \to a} f(x)$ exists
3. $\lim\limits_{x \to a} f(x) = f(a)$

is violated, *a* is called a point of discontinuity of the function.

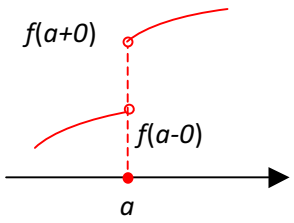
It is clear, that the condition of continuity is violated if one of the following cases takes place:

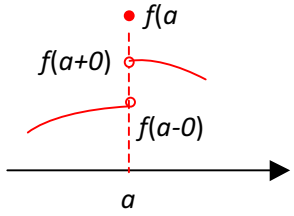
I. The limit $\lim\limits_{x \to a} f(x)$ exists but does not coincide with $f(a)$ i.e. $\exists \lim\limits_{x \to a} f(x) \neq f(a)$.

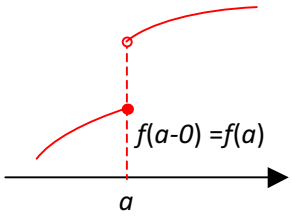


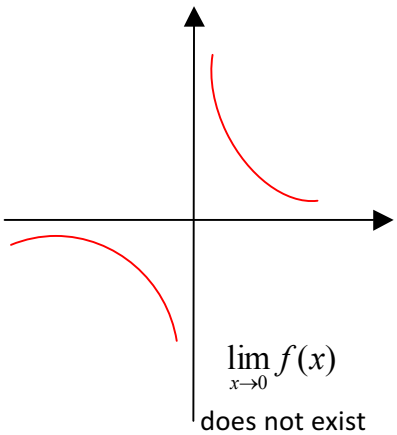


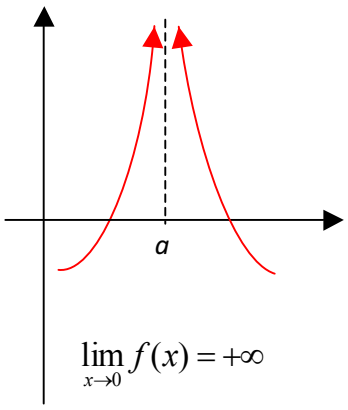
II. The limit entering into condition of continuity is infinite or does not exist at all (a function must not necessarily be defined at its point of discontinuity).

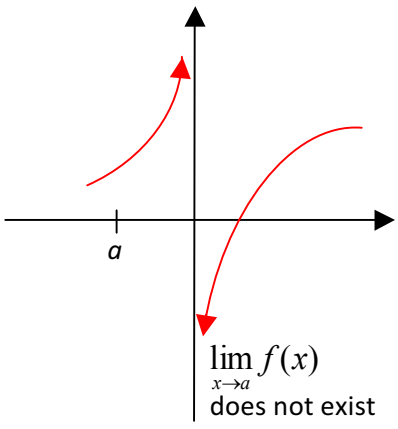












Let us state the following definition.

Definition. If a function $f(x)$ possesses the left-hand and the right-hand limits at its point of discontinuity *a*

i.e. $\exists f(a-0)$ and $\exists f(a+0)$ this point *a* is called point of discontinuity of the fist kind.

Definition. All the points of discontinuity which are not the points of discontinuity of the first kind are referred to as the points of discontinuity of the second kind.

If the limit $\lim\limits_{x\to a} f(x)$ exists but the function value *f*(*a*) is different (i.e. $\lim\limits_{x\to a} f(x) \neq f(a)$) or function *f* is even not defined at point *a* we say that function *f* has **a removable discontinuity** at *a*. If the limit $\lim\limits_{x\to a} f(x)$ exist but the function *f* is not defined at point *a*, we say that function *f* has **a gap** at *a*.

A function f has **a finite jump** at *a* if both one-sided limits $f(a-0)$ and $f(a+0)$ exist and they are different. The magnitude $|f(a+0)-f(a-0)|$ is called a jump of the function at *a*.

Now we state the following.

Theorem. For a function *f* to be continuous at a point *x*=*a* it is necessary and sufficient that $f(a-0)=f(a+0)=f(a)$.

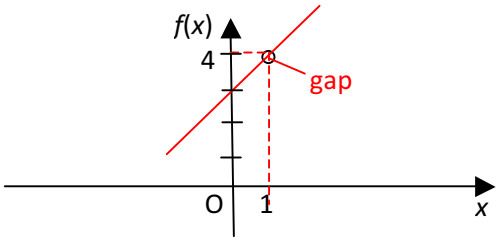
Example 1. Let function *f* be given by formula

$$f(x)=\frac{x^2+2x-3}{x-1}$$

In this case, *f* is not defined at *x*=1, but $\lim_{x\to 1} f(x)=4$. Thus, function *f* has a gap at point *x*=1. This is a removable discontinuity, since we can define a function *g* with

$$g(x)=\begin{cases} f(x) & \text{for } x\neq 1 \\ 4 & \text{for } x=1 \end{cases}$$

Which is continuous at point *x*=1



Example 2. Consider a function *f* given by the formula

$$f(x)=\frac{|x|}{x}=\begin{cases} 1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{cases}$$

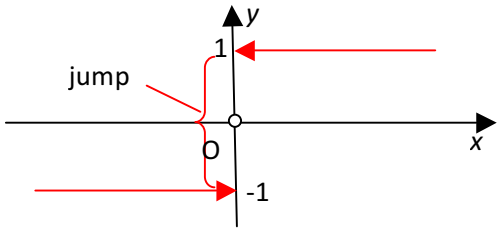
It is clear, that f is not defined at x=0.

We have $f(a-0)=\lim_{x\to-0} f(x)=-1$

$$f(a+0)=\lim_{x\to+0} f(x)=1$$

It follows that the limit $\lim_{x\to 0} f(x)$, does exist.

Thus, function f has a jump at x=0 (discontinuity of the first kind) and the jump is equal to $f(0+0)-f(0-0)=2$.



Example 3. Consider a function f given by the formula

$$f(x)=\frac{\sin x}{|x|}$$

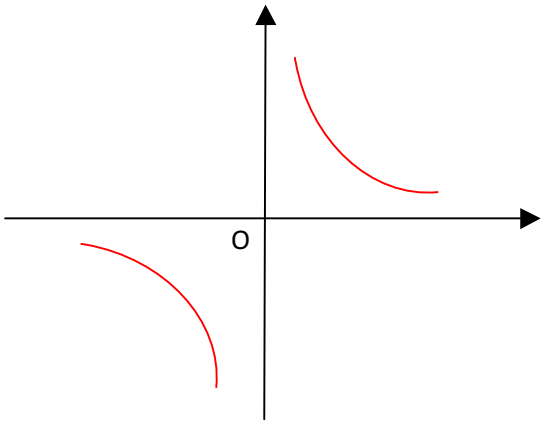
We have

$$f(+0)=\lim_{x\to+0} f(x)=\lim_{x\to+0}\frac{\sin x}{|x|}=\lim_{x\to+0}\frac{\sin x}{x}=1$$

$$f(-0)=\lim_{x\to-0}f(x)=\lim_{x\to-0}\frac{\sin x}{|x|}=\lim_{x\to-0}\frac{\sin x}{-x}=-1$$

Hence, function f has a jump at x=0 (discontinuity of the first kind) and the jump is equal to 2.

Example 4. The function $f(x)=\frac{1}{x}$ is not defined at x=0 and has the discontinuity of the second kind at x=0 since $\lim_{x\to0}\frac{1}{x}$ does not exist.



Lecture 5

Basic properties of functions continuous on a closed interval

Theorem 1. (Weierstrass theorem)

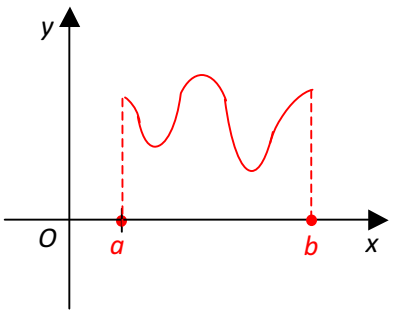
A continuous function on a closed interval is bounded.

i.e. f is continuous on [a,b] $\Rightarrow f$ is bounded on [a,b]

Theorem 2. (Weierstrass theorem)

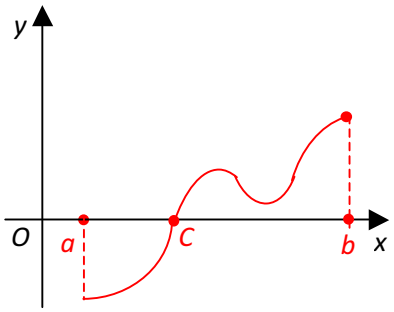
A continuous function on a closed interval attains its bounds.

In other words, this theorem means that if a function is continuous on a closed interval there exist at least one point at which the function assumes the



greatest value and at least one point at which it assumes the least value on that interval.

Theorem 3. (Bolrano-Cauchy theorem)

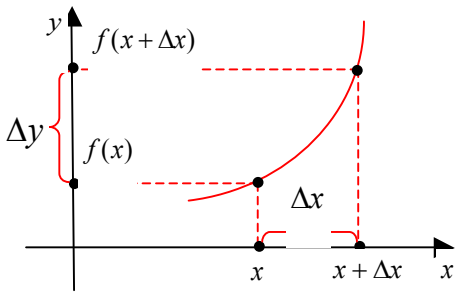


If a function *f* is continuous on a closed interval and assumes value of different signs at its end points (i.e. $f(a)\cdot f(b)<0$) there exists at least one point lying inside the interval at which the function turns into zero.

Lecture 6

Derivative

The concept of derivative is at the core of Mathematics. Let a function $y=f(x)$ be given. If the independent variable *x* is given an increment Δx, the function *y* receives the corresponding increment $\Delta y=f(x+\Delta x)-f(x).$



Definition. The limit of the ratio of the increment of a given function $y=f(x)$to the increment of the independent variable as the latter tends to zero is called the derivative of that function (provided that this limit exists):

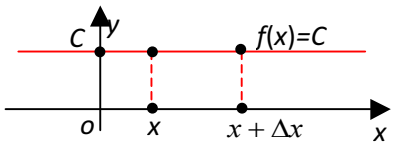
$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Notation. The derivative of a function with respect to a variable *x* is denoted either $f'(x)$ or $\frac{df}{dx}$ (read '*f* dash *x*' or '*f* prime *x*')

Definition. The operation of finding a derivative is called differentiation.

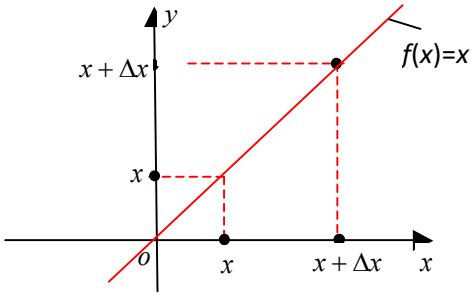
Derivatives of some functions

Let us find the derivatives of some elementary functions.



1) $f(x) = C$, where C is an arbitrary constant.

It is clear that $\Delta y = f(x + \Delta x) - f(x) = C - C = 0$



Therefore

$$f'(x) = (C)' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

Thus,

That is, the derivative of a constant is equal to zero.

2) $f(x) = x$

We have $\Delta y = f(x + \Delta x) - f(x) = x + \Delta x - x = \Delta x$

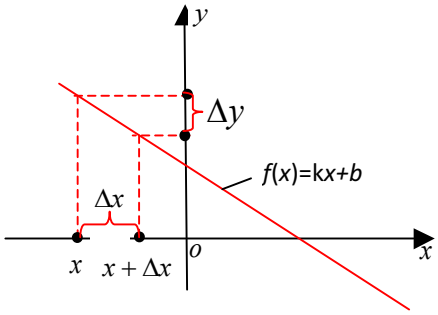
Hence,

$$f'(x)=(x)'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=\lim_{\Delta x\to 0}1=1$$

Thus, $(x)'=1$

That is, the derivative of an independent variable is equal to one.

3) Consider a linear function $f(x)=kx+b$ where *k* end *b* are constant. We can find an expression for Δy



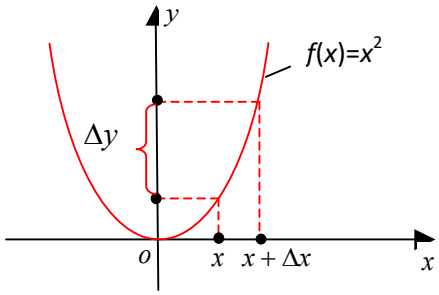
$$\Delta y=f(x+\Delta x)-f(x)=k(x+\Delta x)+b-kx-b=k\Delta x.$$

It follows that

$$f'(x)=(kx+b)'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{k\Delta x}{\Delta x}=\lim_{\Delta x\to 0}k=k$$

Thus, the derivative of a linear function $f(x)=kx+b$ is equal to *k*, that is

$$(kx+b)'=k$$



4) $f(x)=x^2$

We have

$$\Delta y=f(x+\Delta x)-f(x)=(x+\Delta x)^2-x^2=2x\cdot\Delta x+(\Delta x)^2$$

Therefore

$$f'(x)=(x^2)'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

Thus, the derivative of $f(x)=x^2$ is equal to 2*x* that is $(x^2)'=2x$

5) Now let us consider a power function $f(x)=x^n$ for $n\in N$ where N is a set of all positive integers and prove that its derivative is equal to nx^{n-1}

That is $(x^n)^{/}=nx^{n-1}$

To this end, using the well-known Newton's binomial formula we find an expression for Δy

$$\Delta y=f(x+\Delta x)-f(x)=(x+\Delta x)^n-x^n=x^n+nx^{n-1}\cdot\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots+$$
$$+(\Delta x)^n-x^n=\Delta x\left(nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}\Delta x+\ldots+(\Delta x)^{n-1}\right)$$

It follows that

$$\frac{\Delta y}{\Delta x}=nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}\Delta x+\ldots+(\Delta x)^{n-1}$$

It is clear, that the first term on the right-hand side of the last equality does not depend on Δx and the others tend to zero as $\Delta x\to 0$. Therefore,

$$f^{/}(x)=(x^n)^{/}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=nx^{n-1}$$

Thus, for every positive integer n, the power function $f(x)=x^n$ has the derivative nx^{n-1}, which is what we wished to prove. It can be shown that the formula $(x^n)^{/}=nx^{n-1}$ is valid when n is an arbitrary real number.

6) Now let us consider a trigonometric function $f(x)=\sin x$ and prove that the derivative of $\sin x$ is equal to $\cos x$, that is

$$(\sin x)' = \cos x$$

First, we find an expression for Δy. We have

$$\Delta y = f(x+\Delta x) - f(x) = \sin(x+\Delta x) - \sin x = 2\cdot \sin\frac{\Delta x}{2}\cdot \cos\left(x+\frac{\Delta x}{2}\right).$$

Therefore,

$$f'(x) = (\sin x)' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\cdot \sin\frac{\Delta x}{2}\cdot \cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot \lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right) = \cos x$$

Thus, $(\sin x)' = \cos x$,

which is what we had to prove.

7) Let us consider another trigonometric function $f(x) = \cos x$ and prove that the derivative of $\cos x$ is equal to $-\sin x$ that is

$(\cos x)' = -\sin x$

Since $\Delta y = f(x+\Delta x) - f(x) = \cos(x+\Delta x) - \cos x = -2\cdot \sin\frac{\Delta x}{2}\cdot \sin\left(x+\frac{\Delta x}{2}\right)$

We have

$$f'(x) = (\cos x)' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\cdot \sin\frac{\Delta x}{2}\cdot \sin\left(x+\frac{\Delta x}{2}\right)}{\Delta x} = -\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot \lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right) = -\sin x.$$

Thus, $(\cos x)' = -\sin x$

and we get what we had to prove.

8) $f(x) = a^x$

We have $\Delta y = a^{x+\Delta x} - a^x = a^x(a^{\Delta x} - 1)$

It follows that

$$f'(x) = (a^x)' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x} = a^x \cdot \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^x \cdot \ln a$$

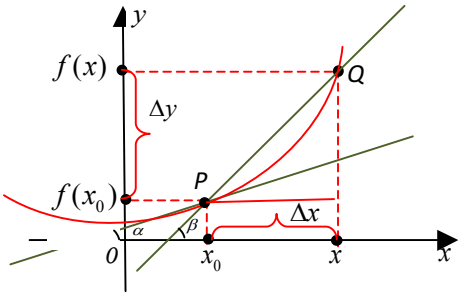
Thus, the derivative of a^x is equal to $a^x \ln a$,

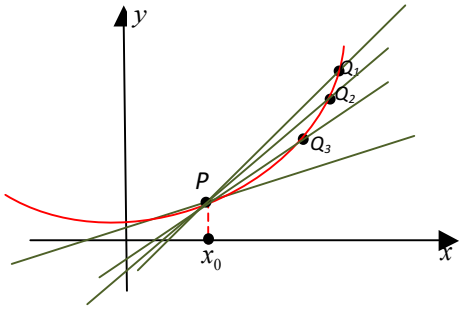
That is, $(a^x)' = a^x \ln a$.

In particular, we have $(e^x)' = e^x$

Geometrical meaning of the derivative

Let us consider the graph of a function $y = f(x)$ and a point $P(x_0, f(x_0))$ on it. Next we will consider a nearby point $Q(x, y)$ which is also on the graph. Let us join the point *Q* to the point *P* with a straight line which is called a secant.





We can calculate the gradient (slope) of the secant.

Putting $\Delta x = x - x_0$ and $\Delta y = f(x) - f(x_0)$, we see that $\frac{\Delta y}{\Delta x} = \tan\beta$, i.e. the ratio is equal to the gradient of the secant. If $\Delta x \to 0$ the point *Q* gets closer and closer to the point *P* . In other words, if $\Delta x \to 0$ then the secant turns round the point P and tends to the limiting position.

Now we can state the definition.

Definition. The tangent to the graph of a function at its given point is the straight line occupying the limiting position of the secant when the points of intersection merge.

Therefore $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \tan\alpha$.

That is the geometrical meaning of the derivative of a function is that it is equal to the slope of the tangent.

Theorem.

The value of the derivative $f'(x_0)$ is equal to the slope of the tangent to the graph of the function $y = f(x)$ at the point with abscissa x_0:

$f'(x_0) = \tan\alpha$,

where α is an angle formed by a tangent and the positive direction of $x-axis$.

Equation of a tangent

We have if a function $y = f(x)$ has a derivative $f'(x_0)$ at a given point x_0 then there exist a tangent drawn to the graph of the function at the point $P(x_0; f(x_0))$ with the gradient $f'(x_0)$.

We now look at how to find the equation of this tangent. The tangent is a straight line with the gradient $f'(x_0)$, so its equation is given by $y = f'(x_0)x + b$. Since the tangent passes through the point *P* we have

$$f(x_0) = f'(x_0)x_0 + b$$

hence

$$b = f(x_0) - f'(x_0)x_0$$

It follows that the equation of a tangent has the form

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Normal. Equation of a normal

The normal to a curve at a given point is a straight line which crosses the curve at that point and is perpendicular to the tangent at that point.

Since the tangent and normal are perpendicular to each other, if the gradient of the tangent is $f^{/}(x_0)$, then the gradient of the normal is $-\frac{1}{f^{/}(x_0)}$. Therefore the equation of the normal has the form

$$y = f(x_0) - \frac{1}{f^{/}(x_0)}(x - x_0)$$

or

$$y - f(x_0) = -\frac{1}{f^{/}(x_0)}(x - x_0)$$

Basic differentiation rules

Differentiation and Arithmetic Operations

Suppose the functions $u = u(x)$ and $v = v(x)$ have derivatives $u^{/}(x)$ and $v^{/}(x)$.

Theorem 1. The derivative of a sum of two functions is equal to the sum of the derivatives of that functions.

$$(u+v)^{/} = u^{/} + v^{/}$$

Proof. Let y be a function represented as the sum of given functions u and v:

$y = u + v$

If $y(x) = u(x) + v(x)$ then $y(x + \Delta x) = u(x + \Delta x) + v(x + \Delta x)$.

If the independent variable *x* is given an increment Δx the functions u, v gain the corresponding increments Δu, Δv and the function *y* the increment Δy. It is clear that

$$\Delta y = y(x+\Delta x) - y(x) = [u(x+\Delta x) + v(x+\Delta x)] - [u(x) + v(x)] =$$
$$[u(x+\Delta x) - u(x)] + [v(x+\Delta x) - v(x)] = \Delta u + \Delta v$$

that is the increment of a sum is equal to the sum of the increments:

$$\Delta y = \Delta(u+v) = \Delta u + \Delta v$$

On dividing the latter relation by Δx we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Now, passing to the limit as $\Delta x \to 0$ we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' + v'$$

Thus, $y' = (u' + v') = u' + v'$, which is what we had to prove.

Similarly can be proved the following

Theorem 2. The derivative of a difference is equal to the difference of the derivatives

$$(u' - v') = u' - v'$$

Now let us prove the following.

Theorem 3. The derivative of the product of two functions is equal to the sum of the product of the derivative of the first function by the second function and the product of the derivative of the second function by the first function.

$$(uv)' = u'v + v'u$$

Proof**.** Let a function *y* be represented as a product of two functions *u* and *v*: *y=uv*

Then

$$\Delta y = y(x+\Delta x) - y(x) = u(x+\Delta x)v(x+\Delta x) - u(x)v(x) =$$
$$= [u(x+\Delta x) - u(x)]v(x+\Delta x) + [v(x+\Delta x) - v(x)]u(x) =$$
$$\Delta u \cdot v(x+\Delta x) + \Delta v \cdot u(x) = \Delta u \cdot [v(x) + \Delta v] + \Delta v \cdot u(x) = \Delta u \cdot v(x) + \Delta v \cdot \cdot u(x) + \Delta u \cdot \Delta v$$

That is

$$\Delta y = \Delta u \cdot v + \Delta v \cdot u + \Delta u \cdot \Delta v$$

On dividing the latter relation by Δx we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \cdot v + \frac{\Delta v}{\Delta x} \cdot u + \frac{\Delta u}{\Delta x} \cdot \frac{\Delta v}{\Delta x} \Delta x$$

and passing to the limit as $\Delta x \to 0$ we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot v + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \cdot u + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \frac{\Delta v}{\Delta x} \cdot \Delta x =$$
$$= v \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x =$$
$$= v \cdot u' + u \cdot v' + u' \cdot v' \cdot 0 = u' \cdot v + v' \cdot u$$

Thus, $y' = (uv)' = u'v + v'u$

which is what we wished to prove.

Corollary. A constant factor can be taken outside the derivative sign, i,e.

$y' = (Cu)' = C \cdot u$

where C – constant.

Theorem 4. The derivative of the quotient of two functions is equal to the fraction whose denominator is equal to the square of the divisor and the numerator is equal to the difference between the product of the derivative of the dividend by the division and the product of the dividend by the derivative of the divisor

$$\left(\frac{u}{v}\right) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Proof. Let $y = \frac{u}{v}$. Then

$$\Delta y = y(x + \Delta x) - y(x) = \frac{u(x + \Delta x)}{v(x + \Delta x)} - \frac{u(x)}{v(x)} = \frac{u(x) + \Delta u}{v(x) + \Delta v} - \frac{u(x)}{v(x)} = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{\Delta u \cdot v - u \cdot \Delta v}{v \cdot (v + \Delta v)}$$

That is, $\Delta y = \frac{\Delta u \cdot v - u \cdot \Delta v}{v \cdot (v + \Delta v)}$

On dividing the latter relation by Δx we get

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta u}{\Delta x} \cdot v - u \frac{\Delta v}{\Delta x}}{v \cdot \left(v + \frac{\Delta v}{\Delta x} \cdot \Delta x\right)}$$

and passing to the limit as $\Delta x \to 0$ we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta u}{\Delta x} \cdot v - u \cdot \frac{\Delta v}{\Delta x}}{v\left(v + \frac{\Delta v}{\Delta x} \cdot \Delta x\right)} = \frac{u'v - uv'}{v(v + v' \cdot 0)} = \frac{u'v - uv'}{v^2}$$

Thus, $y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

which is what we had to prove.

Derivative of a Composite Function

Let $y = f(u)$ and $u = g(x)$, then the composite function of *g* and *f* is

$(fog)(x) = f(g(x))$

Theorem. The derivative of a composite function $y = f(g(x))$ can be found by the formula

$$y' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$

or $y' = f'(u) \cdot u' = f'(u) \cdot g'(x)$

that is, the derivative of a composite function $y = f(g(x))$ is equal to the product of the derivative of the given function $f'(u)$ with respect to the intermediate argument *u* by the derivative of this argument $u' = g'(x)$ with respect to the independent variable *x*.

Proof. If *x* receives an increment Δx then the intermediate variable *u* receives an increment Δu and therefore *y* receives an increment Δy too. We have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \qquad (1)$$

Now let $\Delta x \to 0$. Then $\frac{\Delta u}{\Delta x} \to u'_x$, i.e. $\lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = g'(x)$

and hence $\Delta u = \frac{\Delta u}{\Delta x} \cdot \Delta x \to u'_x \cdot 0 = 0$ i.e. $\Delta u \to 0$.

Therefore $\frac{\Delta y}{\Delta u} \to y'_u$, i.e. $\lim\limits_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(x)$

Now, passing to the limit as $\Delta x \to 0$ in formula (1) and taking into account that if $\Delta x \to 0$ then Δu also tends to zero, $\Delta u \to 0$, we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta u \to 0} \frac{\Delta u}{\Delta x} = f'(u) \cdot g'(x)$$

which is what we wished to prove

Derivative of an Inverse Function

Suppose the function $y = f(x)$ has the inverse $x = f^{-1}(y)$. We will prove that if the derivative of the original function $y = f(x)$ is known it is easy to determine the derivative of the inverse function $x = f^{-1}(y)$.

Theorem. The derivative of the inverse function can be found by the formula

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}} \quad \text{or} \quad x'_y = \frac{1}{y'_x}$$

Proof. Indeed, we have

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$$

and passing to the limit as $\Delta x \to 0$ and taking into account that if $\Delta x \to 0$ then Δy also tends to zero, $\Delta y \to 0$, we obtain

$$x'_y = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim\limits_{\Delta x \to 0} \frac{\Delta x}{\Delta y}} = \frac{1}{y'_x}$$

which is what we had to prove.

Derivative of a Function Specified Parametrically

Now we will study a method of how to find the derivative of a function specified by parametric equations.

Let *y* as function of *x* be specified parametrically by equations

$$x = \varphi(t), \quad y = \psi(t).$$

Theorem. The derivative of a function specified parametrically $x = \varphi(t)$, $y = \psi(t)$ can be found by the formula

$$y'_x = \frac{\psi'(t)}{\varphi'(t)} \quad \text{or} \quad y'_x = \frac{y'_t}{x'_t}$$

Proof. Indeed, according to the differentiation rule for composite functions we have

$$y'_x = \psi'(t) \cdot t'_x$$

And the derivative t'_x is found with the aid of the differentiation rule for an inverse function

$$t'_x = \frac{1}{x'_t} = \frac{1}{\varphi'(t)}$$

Therefore, finally, we have

$$y'_x = \psi'(t) \cdot \frac{1}{\varphi'(t)} = \frac{\psi'(t)}{\varphi'(t)} = \frac{y'_t}{x'_t}$$

which is what we wished to prove.

Derivatives of inverse trigonometric functions

1. $y = \arcsin x, \quad x = \sin y, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}, -1 \le x \le 1$

Applying the differentiation rule for an inverse function, we get

$$y'_x = (\arcsin x)' = \frac{1}{x'_y} = \frac{1}{\cos y}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then $\cos y > 0$ and hence, $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$. Thus, finally, we get

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

2. $y = \arccos x, \quad x = \cos y, \quad 0 \le y \le \pi, \quad -1 \le x \le 1$

Similarly, we have

$$y'_x = (\arccos x)' = \frac{1}{x'_y} = \frac{1}{\sin y}$$

Since $0 \le y \le \pi$, then $\sin y > 0$ and therefore,

$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$. Thus, we obtain

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

3. $y = \arctan x, \quad x = \tan y, \; -\frac{\pi}{2} \le y \le \frac{\pi}{2}, -\infty \le x \le +\infty$

Using the formulas for the derivatives of an inverse function and of the tangent we obtain

$$y'_x = (\arctan x)' = \frac{1}{x'_y} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+x^2}$$

Thus,

$$(\arctan x)' = \frac{1}{1+x^2}$$

4. $y = arc\cot x, \quad x = \cot y, \; 0 \le y \le \pi, \qquad -\infty \le x \le +\infty$

Similarly,

$$y'_x = (arc\cot x)' = \frac{1}{x'_y} = \frac{1}{\frac{1}{\sin^2 y}} = -\sin^2 y = -\frac{1}{1+\cot^2 y} = \frac{1}{1+x^2}$$

Thus,

$$(arc\cot x)' = -\frac{1}{1+x^2}$$

Derivative of a logarithmic Function

We have $y = \log_a x,\ x = a^y,\ \ a > 0, a \neq 1,\ x > 0,\ -\infty < y < +\infty$

Applying the differentiation rule for on inverse function and the formula for the derivative of an exponential function we get

$$y'_x = (\log_a x)' = \frac{1}{x'_y} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

Thus,

$$(\log_a x)' = \frac{1}{x \ln a}$$

In particular, when $a = e$ we have

$$(\ln x)' = \frac{1}{x}$$

Derivative of a Power Function

According to the formula of the derivative of a composite function we have

$$(x^n)' = \left[(e^{\ln x})^n\right]' = \left[e^{n\ln x}\right]' = e^{n\ln x}\cdot n\frac{1}{x} = x^n \cdot n\frac{1}{x} = nx^{n-1}$$

Thus,

$(x^n)' = nx^{n-1}$

Table of Derivatives

Function	Derivative
$y = f(x)$	$y' = f'(x)$
C	0
x	1
x^2	$2x$
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Function	Derivative
$y = f(x)$	$y' = f'(x)$
a^x	$a^x \ln a$
e^x	e^x
$\log_a x$	$\frac{1}{x\ln a}$
$\ln x$	$\frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$

$arc\cot x$	$-\frac{1}{1+x^2}$

Differential

Suppose a function $y = f(x)$ has the derivative at some point *x*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

This implies

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha$$

Where $\alpha = \alpha(\Delta x)$ is an infinitesimal as $\Delta x \to 0$, i.e $\lim_{\Delta x \to 0} \alpha(\Delta x) = 0$

If follows that

$$\Delta y = f'(x)\Delta x + \alpha \cdot \Delta x$$

Definition. A linear function $f'(x)\Delta x$ ($f'(x)$-constant) is called **the differential** of the function $y = f(x)$ and is denoted dy or $df(x)$.

In other words

$$dy = df(x) = y' \cdot \Delta x = f'(x) \cdot \Delta x = f'(x)dx \quad (dx = \Delta x)$$

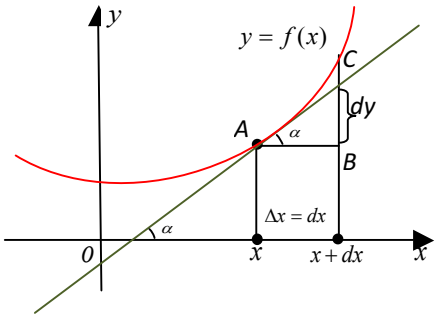
Thus, the differential of the function is the product

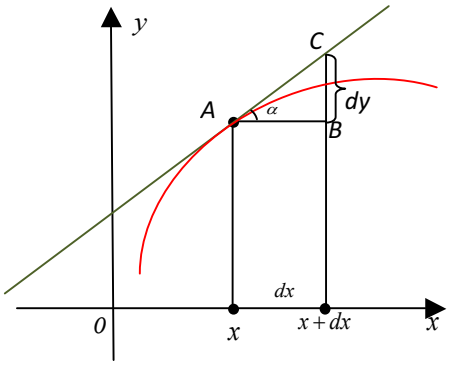
$$dy = f'(x)dx$$

It is clear, that the existence of the differential at a point *x* is equivalent to the existence of the derivative $f'(x)$ at that point *x*.

Geometrical Meaning of the Differential

Let us discuss the geometrical meaning of the differential of a function $y = f(x)$. Since $f'(x) = \tan\alpha$ the differential $dy = f'(x)dx$ is equal to the length of the line segment, BC.





Thus, the geometrical meaning of the differential of a function: the differential is equal to the increment of the ordinate of the tangent.

Differentiability

Definition. A function $y = f(x)$ is **said to be differentiable at a point** *x* if it possesses the differential (or the derivative) at that point.

In short, a function which has the differential (or the derivative) is called **differentiable**.

Theorem. If a function $y = f(x)$ is differentiable at some point, it is continuous at that point.

Proof. Let $y = f(x)$ be differentiable at a point *x*. It means that

$$\Delta y = f(x+\Delta x) - f(x) = f'(x)\cdot\Delta x + \alpha\cdot\Delta x, \text{ where } \lim_{\Delta x\to 0}\alpha = 0$$

Now, passing to the limit as $\Delta x \to 0$, we get

$$\lim_{\Delta x\to 0}\Delta y = \lim_{\Delta x\to 0}\left[f(x+\Delta x) - f(x)\right] = f'(x)\cdot\lim_{\Delta x\to 0}\Delta x + \lim_{\Delta x\to 0}\alpha\cdot\lim_{\Delta x\to 0}\Delta x = 0$$

This yields

$$\lim_{\Delta x\to 0}\Delta y = 0 \text{ or } \lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$$

That is, the function $y = f(x)$ is continuous at *x*. The theorem is proved.

Thus, to put it briefly, if a function $y = f(x)$ is differentiable it is continuous. It should be noted that the converse statement is not true since there are examples of continuous functions which do not possess derivatives at some points.

For instance, the function $y = |x|$ is continuous throughout *Ox* but has no derivative at the point *x*=0

Properties of Differential

The differential of a function is obtained by multiplying its derivative by the differential of the argument and therefore each property of the derivative obviously implies the corresponding property of the differential.

In short,

$$d(u+v) = du + dv$$

$$d(u-v) = du - dv$$

$$d(uv) = vdu + udv$$

$$d(Cu) = Cdu$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

Derivatives and Differentials of Higher Orders

I. Derivatives of higher orders. Let $y = f(x)$. Then the derivative $y' = f'(x)$ is called the derivative of the first order or the first derivative of the function $f(x)$. In its turn, $f'(x)$ is also a function of x and therefore it is possible to take its derivative which is called the derivative of the second order or the second derivative of the original function:

$$y'' = (y')' = f''(x) .$$

In the same way we define the derivative of the third order (the third derivative):

$$y''' = (y'')' = f'''(x)$$

In the general case, the derivative of the n-th order is defined as

$$y^{(n)} = (y^{(n-1)})' = f'^{(n)}(x) .$$

II. Differentials of higher orders. Let $y = f(x)$. Then

$$dy = f'(x)dx$$

The differential *dy* is called the first differential (the first-order differential). The differential of the second order (the second-order differential) of a function is the first differential of the first differential of the function. In is denoted by d^2y

$$d^2y = d(dy) = d(f'(x)dx) = d(f'(x))dx = (f'(x))'dx \cdot dx = f''(x)(dx)^2 = f''(x)dx^2$$

Similarly, the third differential or the differential of the third order d^3y of a function $y = f(x)$ is the differential of its second differential as a function of *x*.

$$d^3y = d(d^2y) = d(f''(x)dx^2) = d(f''(x))dx^2 = (f''(x))'dx \cdot dx^2 = f'''(x)dx^3$$

In the general case, the differential of the *n* th order of a function $y = f(x)$ is defined as

$$d^n y = d(d^{n-1} y) = f^{(n)}(x) dx^n .$$

Lecture 7

Basic theorems of differential calculus

Rolle's Theorem. If a function $f(x)$ is continuous in a closed interval $[a,b]$, is differentiable at all its interior points and assumes equal values at the end points of the interval: $f(a) = f(b)$, then there is at least one point $x = c \in (a,b)$, i.e. lying inside that interval at which $f'(c) = 0$.

Proof. It is known that a function continuous in a closed interval attains its greatest and least values on that interval. Suppose $M = \max f(x)$, $m = \min f(x)$, then $m \le f(x) \le M$ for all $x \in [a,b]$. There are two possible cases here: $m = M$ and $m < M$. (i.e. $m \ne M$). If $m = M$ then $f(x)$ is constant and hence $f'(x) = 0$ for all $x \in (a,b)$. As a point *c* in this case we can take any point of the interval (a,b).

If $m \ne M$ then from the condition $f(a) = f(b)$ it follows that at least one of the values *m* or *M* is not accepted on the ends. For definiteness, let the function $f(x)$ assumes its greatest value M inside the interval (a,b) at some point $c \in (a,b)$, i.e. $f\ (c) = M$. It follows that $f(c + \Delta x) - f(c) \le 0$ for any Δx , and hence

$$\frac{f(c+\Delta x)-f(c)}{\Delta x}\le 0 \text{ for } \Delta x>0$$

$$\frac{f(c+\Delta x)-f(c)}{\Delta x}\ge 0 \text{ for } \Delta x<0$$

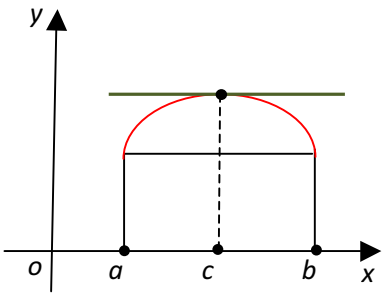
Now, passing to the limit as $\Delta x\to 0$ in the last two inequalities, we get

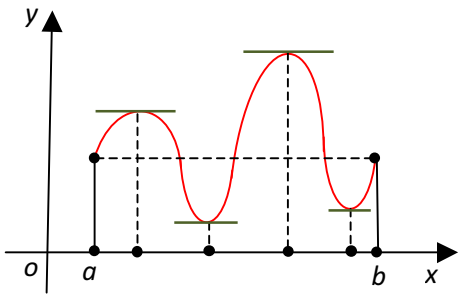
$$\lim_{\Delta x\to +0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c)\le 0 \text{ for } \Delta x>0$$

$$\lim_{\Delta x\to -0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c)\ge 0 \text{ for } \Delta x<0$$

It follows that $f'(c)=0$. Thus, inside the interval (*a*,*b*) there is a point *c* at which the derivative is equal to zero.

Now let us discuss the geometrical meaning of the **Rolle's theorem**.





Rolle's theorem is interpreted geometrically as follows: on the graph of a function continuous in a closed interval and differentiable inside the interval, assuming equal values at the end points of the interval, there is at least one point at which the tangent is parallel to the x-axis.

Lagrange's Theorem. If a function $f(x)$ is continuous in a closed interval $[a,b]$ and is differentiable inside the interval (a,b) there is at least one point $x = c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{(b - a)} = f'(c) .$$

Proof. Let consider the auxiliary function

$F(x) = f(x) - \lambda x$

And determine the number λ so that $F(a) = F(b)$.

We have $f(a) - \lambda a = f(b) - \lambda b$

Then $\lambda = \frac{f(b) - f(a)}{b - a}$.

It is clear that the function $F(x)$ satisfies all the conditions of Rolle's theorem. Indeed, $F(x)$ is continuous in $[a,b]$ and is differentiable in (a,b), and assumes equal values at the end points of the interval: $F(a) = F(b)$.

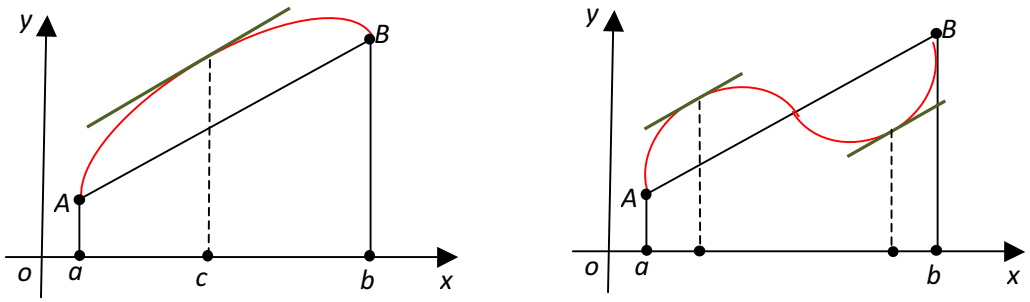
Therefore, by Rolle's theorem, there exist a point $c \in (a,b)$ such that $F'(c) = 0$. But $F'(x) = f'(x) - \lambda$, and hence $F'(c) = f'(c) - \lambda = 0$, i.e. $f'(c) = \lambda$.

Thus, finally, we get

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

which is what we wished to prove.

Not let us discuss the geometrical meaning of the Lagrange's theorem. Let $A=(a,f(a))$and $B=(b,f(b))$be the ends of the graph of a function *f*. let *AB* be a chord connecting the points *A* and *B*.



Lagrange's theorem is interpreted geometrically as follows: On the graph of a function continuous in a closed interval and differentiable inside the interval, there is at least one point at which the tangent is parallel to the chord *AB*.

It follows Lagrange's theorem that

$$f(b)-f(a)=f'(c)(b-a),\quad a<c<b.$$

This relation is known as the formula of finite increments.

Cauchy's Theorem. If $f(x)$ and *g*(*x*) are continuous functions in a closed interval $[a,b]$, differentiable inside the interval (*a*,*b*) and if $g'(x) \neq 0$ for all points $x \in (a,b)$ there exist at least one point $x = c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \ .$$

Before to prove the theorem, it should be noted that $g(b)-g(a) \neq 0$ since, if otherwise, there would exist, according to Rolle's theorem, $c \in (a,b)$ at which $g'(c) = 0$ and hence contradicts the hypothesis.

Proof. Let us consider the auxiliary function

$$F(x) = f(x) - \lambda g(x)$$

and determine the numbe λ so that $F(a) = F(b)$, that is $f(a) - \lambda g(a) = f(b) - \lambda g(b)$.

It follows that $\lambda = \frac{f(b)-f(a)}{g(b)-g(a)}$

It is obviously, the auxiliary function $F(x)$ satisfies all the conditions of Rolle's theorem. Consequently, by Rolle's theorem, there is a point $c \in (a,b)$ such that $F'(c) = 0$, but $F'(x) = f'(x) - \lambda g'(x)$, and hence $F'(c) = f'(c) - \lambda g'(c) = 0$. It follows that

$$\lambda = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Which is what we had to prove.

It should be noted that, in the particular case, when $g(x)=x$. Cauchy's theorem gives same result as Lagrange's theorem.

Lecture 8

L'Hospital - Bernoulli's Rule

Suppose we need to find the limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

as *x* tends to some point.

In many cases, finding the limit is reduced to a formal substitution of the corresponding value instead of argument in the formula that defines the considered function. However, in many cases, this process leads to expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, \ 0\cdot\infty, \ \infty-\infty, \ 1^{\infty}, \ 0^{0}, \ \infty^{0}$. These expressions are called indeterminates because they cannot give us information about the limit.

Indeterminate forms of the type $\frac{0}{0}$

Theorem. Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.

If $f(x)$ and $g(x)$ are differentiable at the point *a* (i.e. their derivatives $f'(a)$ and $g'(a)$ exist) and $g'(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof. Since $f(x)$ and $g(x)$ are differentiable at the point *a* then they are continuous at that point. It means that $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. Consequently, $f(a) = g(a) = 0$ and therefore

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

Now, passing to the limit as $x \to a$ and using the theorem on the limit of a quotient we get

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}.$$

which is what we had to prove.

L'Hospital - Bernoulli's theorem. Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

If the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists then the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ also exist and

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Examples.

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0}\dfrac{(\sin x)'}{(x)'}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=\cos 0=1.$

2. $\lim\limits_{x\to 2}\dfrac{x^2+4x-12}{x^2-6x+8}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 2}\dfrac{(x^2+4x-12)'}{(x^2-6x+8)'}=\lim\limits_{x\to 2}\dfrac{2x+4}{2x-6}=-4.$

3. $\lim\limits_{x\to 1}\dfrac{x^2-1+\ln x}{e^x-e}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 1}\dfrac{(x^2-1+\ln x)'}{(e^x-e)'}=\lim\limits_{x\to 1}\dfrac{2x+\dfrac{1}{x}}{e^x}=\dfrac{3}{e}.$

4. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0}\dfrac{(\ln(1+x))'}{(x)'}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{1+x}}{1}=1.$

It sometimes happens that using L'Hospital - Bernoulli's rule we get a ratio of derivatives which is again an indeterminate form of the $\frac{0}{0}$. In such cases it is possible to apply the rule again and so on.

For example

5. $\lim\limits_{x\to 0}\dfrac{e^{3x}-3x-1}{\sin^2 5x}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0}\dfrac{3e^{3x}-3}{5\sin 10x}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0}\dfrac{9e^{3x}}{50\cos 10x}=\dfrac{9}{50}=0.18$

Here we have applied L'Hospital - Bernoulli's rule two times.

Indeterminate forms of the type $\frac{\infty}{\infty}$

L'Hospital - Bernoulli's Theorem. Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x)=\infty \text{ and } \lim_{x\to a} g(x)=\infty$$

If the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists then the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ also exist and

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Examples.

1. $\lim_{x\to\infty}\frac{\ln x}{x^2}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{(\ln x)'}{(x^2)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=0.$

2. $\lim_{x\to\infty}\frac{\ln x}{x^n}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{\frac{1}{x}}{nx^{n-1}}=\lim_{x\to\infty}\frac{1}{nx^n}=0.$ for $n>0.$

3. $\lim_{x\to+\infty}\frac{x^n}{e^x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{nx^{n-1}}{e^x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{n(n-1)x^{n-2}}{e^x}=\left(\frac{\infty}{\infty}\right)=\ldots=\lim_{x\to+\infty}\frac{n(n-1)(n-2)\ldots 1}{e^x}=0$

If *n* is positive integer.

Here we have applied L'Hospital - Bernoulli's rule n times. Thus, L'Hospital - Bernoulli's rule can be used repeatedly as many times necessary.

L'Hospital - Bernoulli's rule can be applied to some indeterminate forms of other types such as

$0\cdot\infty,\quad \infty-\infty,\ 1^{\infty},\quad \infty^0,\ o^0$

after they are transformed into forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate forms of the type $0\cdot\infty$.

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=\infty.$$

Suppose we need to find the limit $\lim_{x\to a} f(x)g(x)$ which corresponds to the case $0\cdot\infty$. This indeterminate form can be reduced to any of the following

$$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}=\left(\frac{0}{0}\right)$$

$$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}=\left(\frac{\infty}{\infty}\right)$$

and then it is easy to apply the corresponding L'Hospital - Bernoulli's rule.

Examples.

1. $\lim_{x\to 0} x^2\ln x=\lim_{x\to 0}\frac{\ln x}{\frac{1}{x^2}}=\lim_{x\to 0}\frac{(\ln x)'}{\left(\frac{1}{x^2}\right)'}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=-\frac{1}{2}\lim_{x\to 0} x^2=0$

2. $\lim_{x\to 0} x\cdot\cot\pi x=\lim_{x\to 0}\frac{x}{\tan\pi x}=\lim_{x\to 0}\frac{(x)'}{(\tan\pi x)'}=\lim_{x\to 0}\frac{1}{\frac{\pi}{\cos^2\pi x}}=\frac{1}{\pi}\lim_{x\to 0}\cos^2\pi x=\frac{1}{\pi}$

Indeterminate forms of the type $\infty - \infty$

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty.$$

Suppose we want to find the limit $\lim_{x\to a}\left[f(x) - g(x)\right]$ which corresponds the case $\infty - \infty$. We can transform this limit as follows

$$\lim_{x\to a}\left[f(x) - g(x)\right] = \lim_{x\to a}\left[\frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}}\right] = \lim_{x\to a}\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}} = \left(\frac{0}{0}\right).$$

and hence, it is possible to apply L'Hospital - Bernoulli's rule.

Example.

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0}\frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x\to 0}\frac{(e^x - 1 - x)'}{(x(e^x - 1))'} = \lim_{x\to 0}\frac{e^x - 1}{e^x - 1 + xe^x} =$$

$$= \lim_{x\to 0}\frac{(e^x - 1)'}{(e^x - 1 + xe^x)'} = \lim_{x\to 0}\frac{e^x}{e^x(2 + x)} = \frac{1}{2}$$

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\frac{(x\ln x - x + 1)'}{((x-1)\ln x)'} = \lim_{x\to 1}\frac{\ln x}{\ln x + \frac{x-1}{x}} =$$

$$= \lim_{x\to 1}\frac{(\ln x)'}{\left(\ln x + \frac{x-1}{x}\right)'} = \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$$

Indeterminate forms of the type 1^{∞}

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x)=1 \text{ and } \lim_{x\to a} g(x)=\infty.$$

Suppose we want to find the limit $\lim_{x\to a}\left[f(x)\right]^{g(x)}$ which corresponds the case 1^{∞}.

In this case, after obvious transformations, we obtain

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}=\lim_{x\to a}\left\{\left[1+(f(x)-1)\right]^{\frac{1}{f(x)-1}}\right\}^{(f(x)-1)\cdot g(x)}=e^{\lim_{x\to a}\cdot(f(x)-1)\cdot g(x)}$$

Where the $\lim_{x\to a}(f(x)-1)\cdot g(x)$ corresponds to the case $0\cdot\infty$ that was discussed above.

Thus, in the case 1^{∞} we can apply the formula

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to a}\cdot(f(x)-1)\cdot g(x)}$$

Examples.

1. $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{x}=e^{\lim_{x\to\infty}\left(1+\frac{1}{x^2}-1\right)x}=e^{\lim_{x\to\infty}\frac{1}{x}}=e^{0}=1$

2. $\lim_{x\to 0}(1+x)^{\ln x}=(1^{\infty})=e^{\lim_{x\to 0}(1+x-1)\cdot\ln x}=e^{\lim_{x\to 0}x\cdot\ln x}=e^{\lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}}=e^{\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}}=e^{\lim_{x\to 0}(-x)}=e^{0}=1.$

3. $\lim_{x\to 1}x^{\frac{1}{1-x}}=(1^{\infty})=e^{\lim_{x\to 1}(x-1)\cdot\frac{1}{1-x}}=e^{\lim_{x\to 1}(-1)}=e^{-1}.$

Indeterminate forms of the type 0^0

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0.$$

Suppose we want to find the limit $\lim_{x\to a}\left[f(x)\right]^{g(x)}$.

In this case, we have an indeterminate form of the type 0^0. We can transform this limit as follows

$$\lim_{x\to a}\left[f(x)\right]^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim\limits_{x\to a} g(x)\ln f(x)}$$

where the limit $\lim_{x\to a} g(x)\ln f(x)$ corresponds to the case $0\cdot\infty$

Examples.

1. $\lim\limits_{x\to+0} x^x = e^{\lim\limits_{x\to+0} x\ln x} = e^{\lim\limits_{x\to+0}\frac{\ln x}{\frac{1}{x}}} = e^{\lim\limits_{x\to+0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{-\lim\limits_{x\to+0} x} = e^0 = 1$

2. $\lim\limits_{x\to 0} x^{\sin x} = e^{\lim\limits_{x\to+0}\sin x\ln x} = e^{\lim\limits_{x\to+0}\frac{\ln x}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{\sin x}}} = e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}}} = e^{-\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\cdot\tan x\right)} = e^0 = 1$

Lecture 9

Taylor's Formula

The replacement of a function by a simple expression of the type of a polynomial is extremely convenient in many problems of mathematics. The existence of such a polynomial is extremely important since it makes it possible to replace, approximately, the given function by a polynomial.

The computation of the values of this function then reduces to the computation of the values of the polynomial, which can be achieved with the aid of the

simplest arithmetical operations. The evaluation of the accuracy of these approximations is also extremely important for practice.

If the function $f(x)$ possesses the derivative $f'(a)$ at a point *a*, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

If follows that

$$\frac{\Delta y}{\Delta x} = f'(a) + \alpha$$

or

$$\Delta y = f'(a)\Delta x + \alpha \cdot \Delta x$$

That is $f(x) - f(a) = f'(a) \cdot (x - a) + \alpha \cdot (x - a)$

And hence $f(x = f(a) + f'(a) \cdot (x - a) + \alpha \cdot (x - a)$

where $\lim_{x \to a} \alpha = 0$ (i.e. α is an infinitesimal as $x \to a$). In other words, there exists the linear function (i.e. the polynomial of the first degree) .

$$P_1(x) = f(a) + f'(a)(x - a)$$

such that

$$f(x) = P_1(x) + \gamma(x)$$

Where

$\gamma(x) = \alpha(x - a)$ and $\lim_{x \to a} \gamma(x) = 0$

Now dropping the infinitesimal $\gamma(x)$ we get the following approximate formula

$$f(x) \approx P_1(x)$$

with an accuracy $\gamma(x)$ which is an infinitesimal of higher order than $\beta(x) = x - a$, since $\lim\limits_{x \to a} \dfrac{\gamma(x)}{\beta(x)} = 0$. It turns out that such approximate formulas can be made more accurate if we apply derivatives of higher order. This problem is solved by means of Taylor's formula named after Brook Taylor (1685-1731), an English mathematician.

Theorem. If the function $f(x)$ possesses the $(n+1)th$ derivative $f^{n+1}(x)$ (i.e. *n*+1 times differentiable) at all the points of an interval containing the point *a*, then it can be represented as the sum of the polynomial of the n th degree and remainder term Rn:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n, \qquad (1)$$

where $R_n = \alpha(x)(x-a)^n,\ \lim\limits_{x \to a} \alpha(x) = 0$ (Peano's form)

or

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \qquad \xi = a + \theta(x-a),\ 0 < \theta < 1 \ \text{(Lagrange's form)}$$

Formula (1) is called Taylor's formula for *f*(*x*) with the remainder $R_n(x)$.

Proof. Let us first suppose that we are given a polynomial $P_n(x)$ of degree *n*. A polynomial is usually considered as expanded into powers of *x*

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

But it is quite easy to expand it in powers of *x-a* where *a* is an arbitrary number.

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n \qquad (2)$$

The coefficients $c_0, c_1, \ldots, c_n$ here can be found in the following way. First we put $x = a$ and obtain $P_n(a) = c_0$. Then we differentiate the formula (2):

$$P_n'(x) = c_1 + 2c_2(x-a) + \ldots + nc_n(x-a)^{n-1}$$

If now we put here $x = a$ then we get $P_n'(a) = c_1$. Let us differentiate once again:

$$P_n''(x) = 2c_2 + \ldots + n \cdot (n-1)c_n(x-a)^{n-2}$$

which implies $P_n''(a) = 2c_2 = 2!c_2$

Further, in a similar way we derive $P_n'''(a) = 3!c_3$ and so on. Generally, $P_n^{(k)}(a) = k!c_k$, which yields

$$c_k = \frac{P^{(k)}(a)}{k!}$$

Thus, formula (2) can be rewritten in the following form:

$$P_n(x) = P_n(a) + \frac{P_n'(a)}{1!}(x-a) + \frac{P_n''(a)}{2!}(x-a)^2 + \ldots + \frac{P_n^{(n)}(a)}{n!}(x-a)^n$$

If now we take the function $f(x)$ in place of a polynomial $P_n(x)$ then we obtain the polynomial of the form

$$P_n(x) = f(a) + \frac{f^{/}(a)}{1!}(x-a) + \frac{f^{//}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the derivations of which coincide with the derivatives of the corresponding order of the function $f(x)$ at a point *a*:

$$P_n(a) = f(a), \qquad P_n^{/}(a) = f^{/}(a), \ldots, P_n^{(n)}(a) = f^{(n)}(a).$$

Let us denote the difference between $f(x)$ and $P_n(x)$ by $R_n(x)$

$$R_n(x) = f(x) - P_n(x).$$

It is easy see that

$$R_n(a) = R_n^{/}(a) = R_n^{//}(a) = \ldots = R_n^{(n)}(a) = 0$$

Therefore, applying the L'Hospital-Bernoulli's rule *n* times to the limit

$$\lim_{x\to a}\frac{R_n(x)}{(x-a)^n} = \lim_{x\to a}\frac{R_n^{/}(x)}{n(x-a)^{n-1}} = \lim_{x\to a}\frac{R_n^{//}(x)}{n(n-1)(x-a)^{n-2}} = \ldots = \lim_{x\to a}\frac{R_n^{(n)}(x)}{n!} = \frac{R_n^{(n)}(a)}{n!} = 0$$

finally we get $\frac{R_n(x)}{(x-a)^n} = \alpha(x)$, where $\lim_{x\to a}\alpha(x) = = 0$

that is $R_n(x) = \alpha(x)\cdot(x-a)^n = 0((x-a)^n)$.

Definition. The polynomial of the form

$$P_n(x) = f(a) + \frac{f^{/}(a)}{1!}(x-a) + \frac{f^{//}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called Taylor's polynomials of the n th degree.

Thus, the theorem proved above shows that a Taylor polynomial differs from $f(x)$ by a term $R_n(x)$ (the remainder) which is an infinitesimal of the highest order in comparison with all the polynomials of the same degree as $x \to a$. In other words, the Taylor polynomial is a good approximation for the function $f(x)$ as $x \to a$.

$$f(x) \approx P_n(x).$$

Maclaurin's Formula

On particular, when $a = 0$ we obtain the formula

$$f(x) = f(0) + \frac{f'(a)}{1!}x + \frac{f''(a)}{2!}x^2 + \ldots + \frac{f^{(n)}(a)}{n!}x^n + R_n(x)$$

Which is sometimes called Maclaurin's formula, where the remainder $R_n(x)$ has the following forms:

1) Peano's form: $R_n(x) = 0(x^n)$,

$$0(x^n) = \alpha(x) \cdot x^n, \quad \lim_{x \to 0} \alpha(x) = 0$$

2) Lagrange's form: $R_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1.$

Expanding functions by Maclaurin's Formula

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + 0(x^n),$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + 0(x^{2n+2}),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + 0(x^{2n+1}),$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + 0(x^n),$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + 0(x^n).$$

Where $0(x^n)$ is an infinitesimal of higher order than x^n i.e. $0(x^n) = \alpha(x)x^n$, $\lim_{x\to 0}\alpha(x) = 0$.

Lecture 10

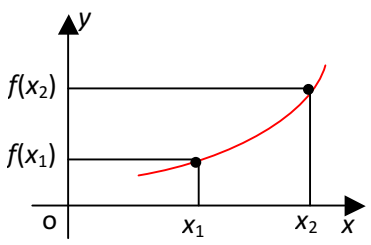
Application of differential calculus to investigation of behaviour of functions

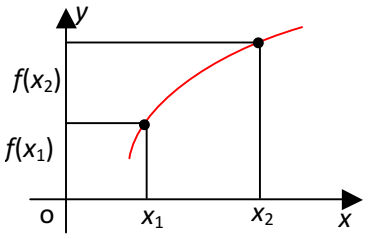
Monotonicity

A function $f(x)$ is called increasing in an interval (*a*,*b*) if for any two points $x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

In other words, a function is increasing if the values of the functions increase as the argument increases.

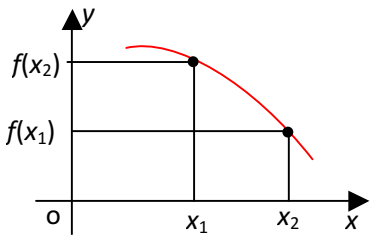


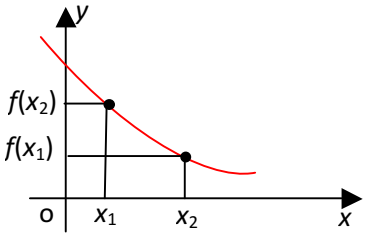


A function $f(x)$ is called decreasing in an interval (*a*,*b*) if for any two points $x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

In other words, a function is decreasing if its values decrease as the argument decreases.





Both increasing and decreasing functions are called monotonic.

Conditions for Monotonicity

Between the character (nature) of the monotonicity of differentiable function and the sigh of its derivative there is a connection stated in the following theorem.

Theorem. Let $f(x)$ be differentiable function in an interval (*a*,*b*). Then

$f'(x) > 0 \Rightarrow f \uparrow$ increases $\Rightarrow f'(x) \geq 0$

$f'(x) < 0 \Rightarrow f \uparrow$ increases $\Rightarrow f'(x) \leq 0$

Proof. Let us prove the first statement, the second is proved similarly.

Suppose $f'(x)$ is positive in an interval (*a*,*b*) , i.e. $f'(x) > 0$ for all $x \in (a,b)$. Now let us take two points $x_1, x_2 \in (a,b)$ and assume that $x_1 < x_2$. According to the Lagrange's theorem we can write

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \ , \quad x_1 < \xi < x_2$$

It is clear that

$f'(x) > 0$ for $x \in (a,b) \Rightarrow f'(\xi) > 0 \Rightarrow \ f'(\xi) \cdot (x_2 - x_1) > 0 \Rightarrow f(x_2) - f(x_1) > 0,$

Thus, we have for any two points $x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

That is $f(x)$ is increasing in (*a*,*b*).

The left part of the first statement has proved. Now let us prove the right part of the first statement. In other words we have to prove that

$$f \uparrow \text{increases} \Rightarrow f'(x) \geq 0$$

Since $f(x)$ is differentiable function in (*a*,*b*), that is $f(x)$ possesses the derivative $f'(x)$ at any point $x \in (a,b)$, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is easy to see that for $\Delta x > 0$ we have

$$x+\Delta x>x \Rightarrow f(x+\Delta x)>f(x) \Rightarrow f(x+\Delta x)-f(x)>0 \Rightarrow$$
$$\Rightarrow \frac{f(x+\Delta x)-f(x)}{\Delta x}>0 \Rightarrow \lim_{\Delta x \to +0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geq 0 \Rightarrow f'(x) \geq 0$$

and for $\Delta x<0$ we have

$$x+\Delta x<x \Rightarrow f(x+\Delta x)<f(x) \Rightarrow f(x+\Delta x)-f(x)<0 \Rightarrow$$
$$\Rightarrow \frac{f(x+\Delta x)-f(x)}{\Delta x}>0 \Rightarrow \lim_{\Delta x \to -0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geq 0 \Rightarrow f'(x) \geq 0$$

Thus, for any case $f'(x) \geq 0$ which is what we had to prove.

Example. Find the intervals of increasing and decreasing of the function $f(x)=x^4-8x^2-3$.

Solution. We find $f'(x)=4x^3-16x$.

It is easy to verify that

$f'(x)>0$ for $x \in (-2,0) \cup (2,+\infty)$ and

$f'(x)<0$ for $x \in (-\infty,-2) \cup (0,2)$

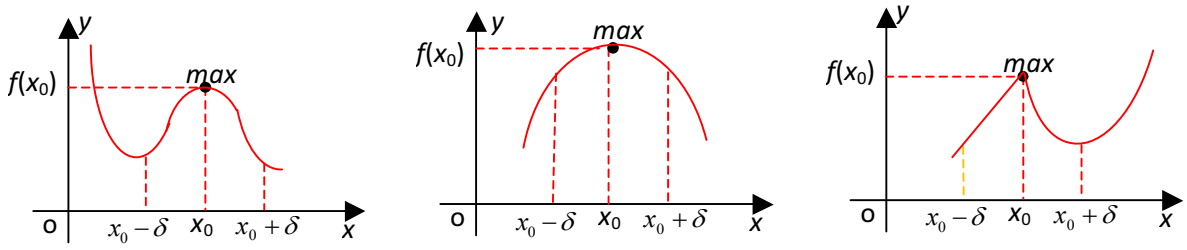
Thus, $f(x)$ increases in $[-2,0] \cup [2,+\infty)$ and

$f(x)$ decreases in $(-\infty,-2] \cup [0,2]$.

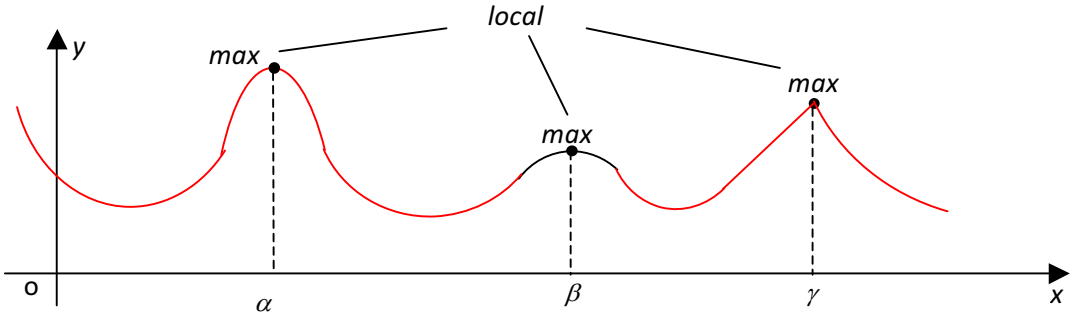
Poins of Extremum. Extrema of Functions

Definition. A point x_0 is called a point of local maximum of the function $f(x)$ if the value $f(x_0)$ is the greatest value of the function $f(x)$ in a neighbourhood of the point x_0.

In other words , it means that there exist a δ- neighbourhood $\delta > 0$ of the point x_0, $U_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ such that for all $x \in U_\delta(x_0)$ $f(x) \le f(x_0)$.



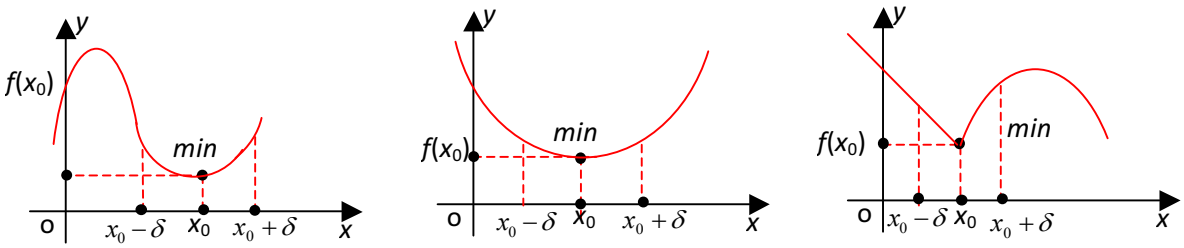
We also say that $f(x_0)$ is a local maximum of the function $f(x)$.



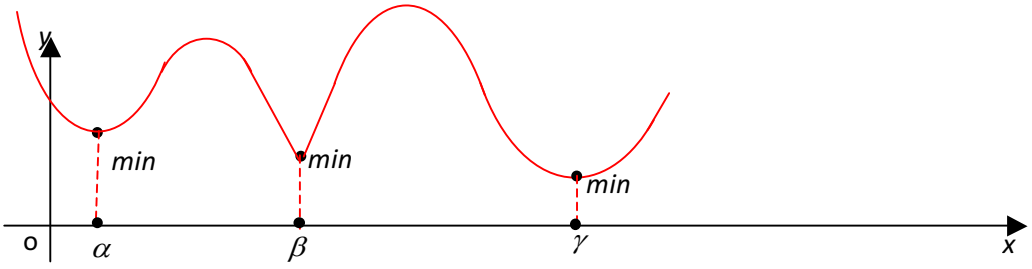
Here, α, β, γ are points of local maximum of the function $f(x)$, and $f(\alpha), f(\beta), f(\gamma)$ are local maximum of the function $f(x)$. The point of local minimum and the local minimum of a function are defined in a similar way.

Definition. A point x_0 is called a point of local minimum of the function $f(x)$, if the value $f(x_0)$ is the least value of the function $f(x)$ in a neighbourhood of the point x_0.

In other words, it means that there exist a δ- neighbourhood, $\delta > 0$, of the point x_0, $U_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ such that for all $x \in U_\delta(x_0)$ $f(x) \geq f(x_0)$.



We also say that $f(x_0)$ is a local minimum of the function $f(x)$.



Here, α, β, γ are points of local minimum of the function $f(x)$, and $f(\alpha), f(\beta), f(\gamma)$ are local minimum of the function $f(x)$.

Definition. The points of local maximum and local maximum are called points of local extremum of the function. The local maximum and the local minimum are called local extrema of the function.

Necessary Condition for Extremum

Theorem. If a function $f(x)$ has an extremum at a point x_0 its derivative at that point either is equal to zero or does not exist.

Proof. For definiteness, let the function $f(x)$ attains a maximum at the point x_0. It means that there exist a neighbourhood $U(x_0)$ of the point x_0 such that for all $x \in U(x_0)$ holds $f(x) \le f(x_0)$

We have for $x < x_0$.

$$\frac{f(x)-f(x_0)}{x-x_0} \ge 0 \Rightarrow f'(x_0) = \lim_{x \to x_0 - 0} \frac{f(x)-f(x_0)}{x-x_0} \ge 0$$

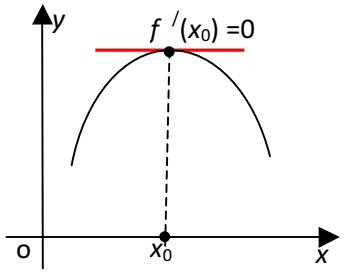
and for $x > x_0$

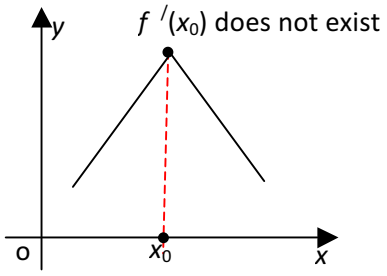
$$\frac{f(x)-f(x_0)}{x-x_0} \le 0 \Rightarrow f'(x_0) = \lim_{x \to x_0 + 0} \frac{f(x)-f(x_0)}{x-x_0} \le 0$$

By the hypothesis, the function $f(x)$ has the derivative $f'(x)$ at the point x_0, and therefore the left-hand and the right-hand limits must coincide. Consequently,

$f'(x_0) \geq 0$ and $f'(x_0) \leq 0 \Rightarrow f'(x_0) = 0$

which is what we had to prove.





The argument is completely similar in the case of a minimum.

Geometrically, the proved theorem states that the tangent drawn to the graph of a function at its highest or lowest points is parallel to *Ox* (i.e.to the axis of abscissas). It should be noted that a function can also have extrema at some of the points where derivative does not exist.

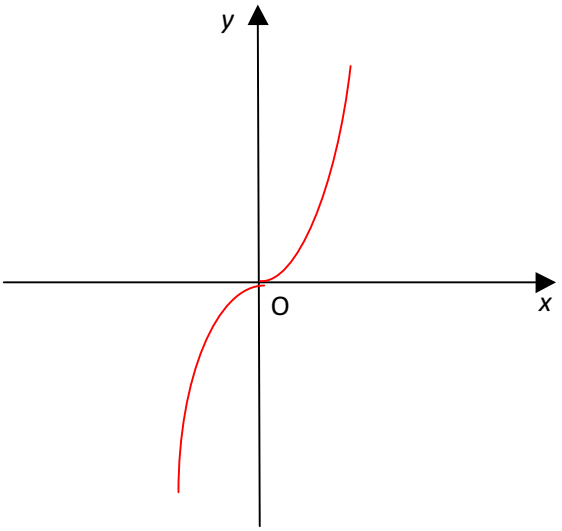
Let us introduce the following concepts.

Definition. A point x_0 at which the derivative is equal to zero is called a stationary point.

x_0 - stationary point $\Leftrightarrow f'(x_0) = 0$

Definition. An interval point x_0 of the domain of definition of the function at which the derivative is equal to zero or does not exist is called a critical point.

It is clear that every stationary point is critical.



The necessary condition for extremum can be restated as follows.

Theorem. If x_0 is a point of extremum then x_0 is a critical point. (i.e. every point extremum is critical).

It should be noted that not every critical point is a point of extremum.

Example. $f(x) = x^3$

$f'(x) = 3x^2 = 0 \Rightarrow x = 0$

It is clear that *x*=0 is a critical point since $f'(0) = 0$.

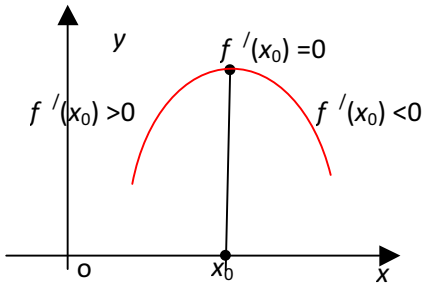
However, *x*=0 is not a point of extremum.

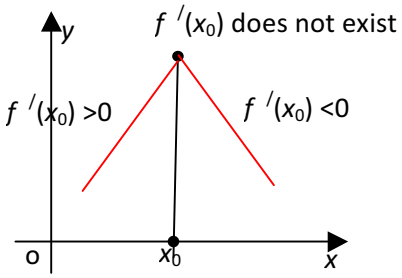
Sufficient Conditions for Extremum

Theorem. (Sufficient Conditions for Extremum in Terms of the First Derivative).

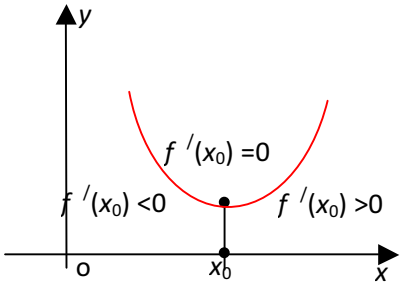
Let $f(x)$ be a continuous function in a neighbourhood of a point x_0, and differentiable in this neighbourhood except possibly at the point x_0 itself. Then:

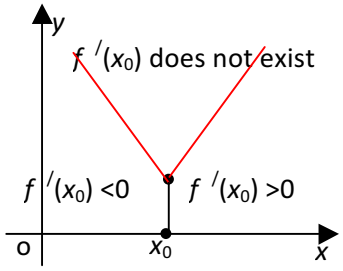
If $f'(x)>0$ for $x<x_0$ and $f'(x)<0$ and $x>x_0$, the point x_0 is a point of maximum





If $f'(x)<0$ for $x<x_0$ and $f'(x)>0$ for $x>x_0$, the point x_0 is a point of minimum





Proof. Let us consider the case: $f'(x)>0$ for $x<x_0$ and $f'(x)<0$ for $x>x_0$. By Lagrange's theorem

$$f(x)-f(x_0)=f'(c)(x-x_0)$$

where either $x < c < x_0$ or $x_0 < c < x$ (i.e. the point c lies in the interval with ends x_0 and x).

If $x < c < x_0$ then $x - x_0 < 0$ and $f'(c) > 0$

and hence

$$f'(c) \cdot (x - x_0) < 0 \Rightarrow f(x) - f(x_0) < 0$$

Therefore,

$$x < c < x_0 \Rightarrow f(x) < f(x_0).$$

If $x_0 < c < x$ then $x - x_0 > 0$ and $f'(c) < 0$

and consequently

$$f'(c) \cdot (x - x_0) < 0 \Rightarrow f(x) - f(x_0) < 0$$

Therefore,

$$x_o < c < x \Rightarrow f(x) < f(x_0).$$

Thus, we get that $f(x) < f(x_0)$ for any case and it means that x_0 is a point maximum.

The other case when $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$ is investigated quite similarly.

The theorem has been proved.

The theorem proved above can be briefly restated as follow:

Theorem. If, as *x* passes through the point x_0, the derivative changes sign the point x_0 is a point of extremum.

Theorem (Sufficient Condition for Extremum in Terms of the Second Derivative).

Let $f'(x_0)=0$ (i.e. x_0 is a stationary point) and $f''(x_0)\neq 0$.

Then:

If $f''(x_0)<0$ the point x_0 is a point of maximum

If $f''(x_0)>0$ the point x_0 is a point of minimum.

Examples.

1. Find points of extremum and extrema of the function

$$f(x)=x^3+3x^2-24x+12$$

Solution. $f'(x)=3x^2+6x-24$

$f'(x)=0\Rightarrow 3x^2+6x-24=0\Rightarrow x^2+2x-8=0\Rightarrow x=-4$ and $x=2$ are stationary points.

Now we calculate the second derivative: $f''(x_0)=6x+6$.

$f''(-4)=-18<0\Rightarrow x=-4$ is a point of maximum and

$f(-4)=92$ is the maximum

$f''(2)=18>0\Rightarrow x=2$ is a point of minimum and

$f(2)=-16$ is the minimum.

2. Find points of extremum and extrema of the function $f(x)=2e^x-5x$

Solution. $f'(x) = 2e^x - 5x$

$f'(x) = 0 \Rightarrow 2e^x - 5x = 0 \Rightarrow e^x = 2.5 \Rightarrow x = \ln 2.5$ is a stationary point

Now we calculate the second derivative: $f''(x) = 2e^x$. Since $2e^x > 0$ for any *x* we get that $x = \ln 2.5$ is a point minimum, and $f(\ln 2.5) = 5 - 5\ln 2.5$ is the minimum.

The Greatest and the Least Values of a Function

Suppose it is necessary to find the greatest and the least values of the function $y = f(x)$ on a closed interval $[a,b]$.

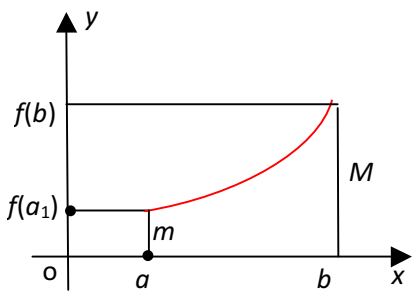
According to well-known Weierstrass's theorem:

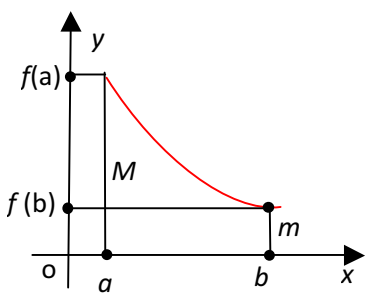
Theorem. A continuous function defined over a closed interval is bounded and attains its least value and its greatest value. In other words, it means that there exist at least one point $c_1 \in [a,b]$ and at least one point $c_2 \in [a,b]$ such that at these points the function attains, respectively, its greatest and least values

$f(c_1) = \max\limits_{a \le x \le b} f(x) = M$ and $f(c_2) = \min\limits_{a \le x \le b} f(x) = m$

We are interested in the question of "How to find these points?". Suppose, at first, a function has no critical points. In that case the function is monotone.

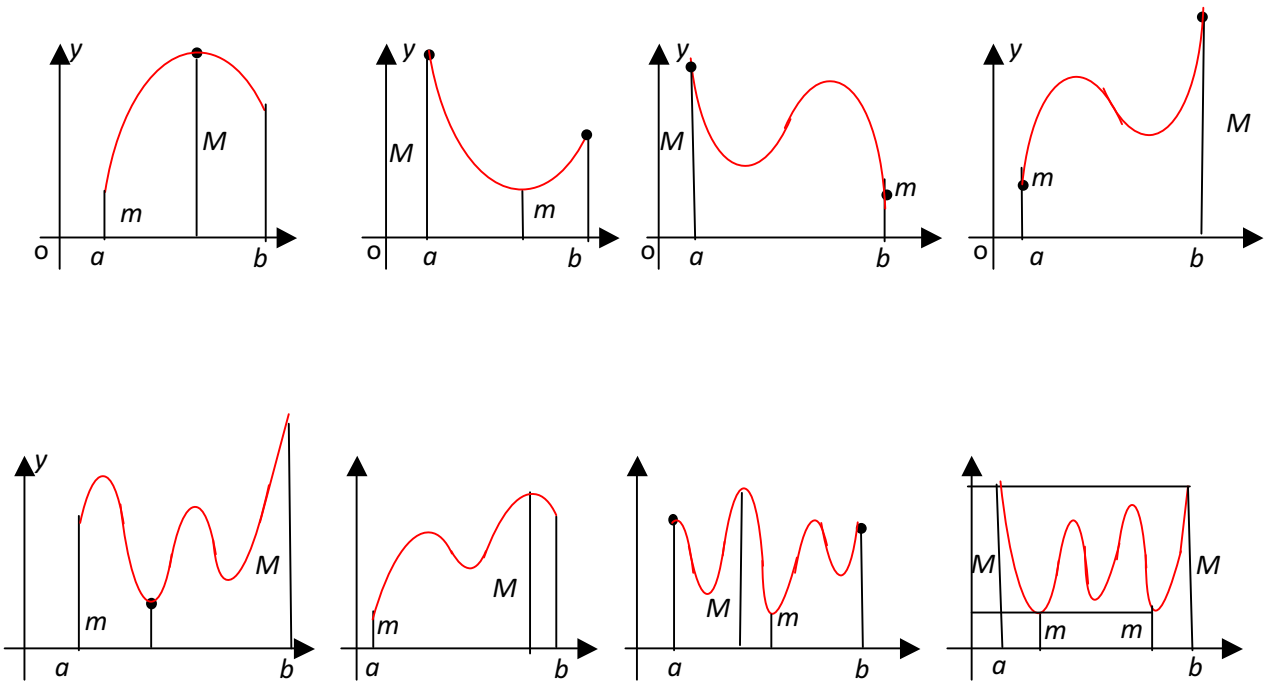
It is clear that when a function $y = f(x)$ is monotone in a closed interval [*a*,*b*]





its greatest value is $M = f(b)$ and the least value is $m = f(a)$ if the function increases and, conversely, the greatest value is $M = f(a)$ and the least value is $m = f(b)$ if the function decreases.

Now suppose a function has a finite number of critical points in [*a*,*b*]. These points divide the closed interval [*a*,*b*] into finite number of segments, in which there are no critical points. Therefore, the greatest and the least values of the function on these segments are assumed at their ends, i.e. at critical points or at points *a* and *b*.



We are now ready to formulate the rule (algorithm). For the case when a function is not only continuous on a closed interval [*a*,*b*], but it has a finite number of critical points in [*a*,*b*], we specify the rule for finding the greatest and the least values.

In order to find the greatest and the least values of a function, $y = f(x)$ on a closed interval [*a*,*b*] and having a finite number of critical points in [*a*,*b*], you need to:

1.Find the critical points belonging to [*a*,*b*].

2. Calculate the values of the function at these critical points and the values of the function at the end points, i.e. $f(a)$ and $f(b)$.

3. Choose from the values obtained the greatest and the least values.

Example.

Find the greatest and the least values of the function $f(x) = 3x^5 - 5x^3$ on the interval [0,2]

Solution. To find the critical points, first find the derivative $f'(x)=15x^4-15x^2$. Solving the equation $f'(x)=0$ we get $x=0,\ x=\pm 1$. It follows that in the closed interval [0,2] there is only one critical point x=1. Recall that a critical point must be internal and cannot be one of the end point of the segment.

Now we calculate $f(0)=0,\ f(1)=-2$ and $f(2)=56$. Comparing these values we get

$\max\limits_{[0,2]} f(x)=f(2)=56$ and $\min\limits_{[0,2]} f(x)=f(1)=-2$

2. Find the greatest and the least values of the function $f(x)=3x\ -x^3$ on the interval [-2,3].

Solution. We find the derivative $f'(x)=3-3x^2$;

$f'(x)=0\ \Rightarrow 3-3x^2=0\Rightarrow x=\pm 1$

Points $x=\pm 1$ are critical points.

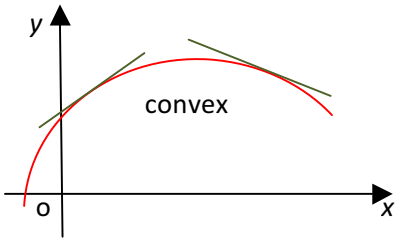
Now we calculate the values of the function at these points: $f(1)=2,\ f(-1)=-2$. We also calculate the values of the function at the and points: $f(-2)=2,\ f(3)=-18$. From the resulting four values we choose the greatest and the least. Thus, we have

$\max\limits_{[-2,3]} f(x)=f(-2)=f(1)=2$ and $\min\limits_{[-2,3]} f(x)=f(3)=-18$

Lecture 11

Convexity and Concavite of Curve

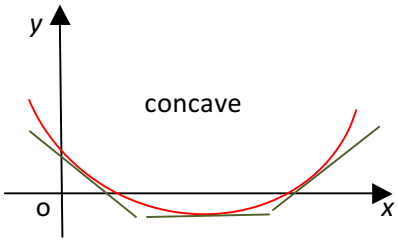
Let a function $y = f(x)$ have a graph of the shape shown below



We see that the graph lies below the tangent drawn at any point.

Definition. The graph of the function $y = f(x)$ is said to be convex (convex up or concave down) on the interval (*a*,*b*) if it lies below the tangent drawn at any point of that interval.

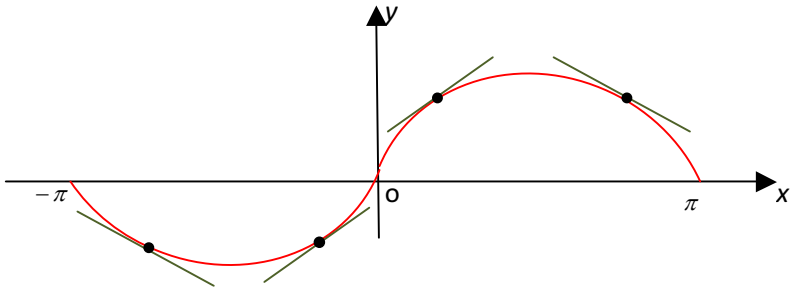
Now let us suppose that the graph of a function $y = f(x)$ has the shape shown below



In this case we see that the graph lies above the tangent drawn at any point.

Definition. The graph of the function $y = f(x)$ is said to be concave (concave up or convex down) on the interval (*a*,*b*) if it lies above the tangent drawn at any point of that interval.

Let us consider in detail the graph of the function $y = \sin x$ (the sinusoid) on the interval $[-\pi, \pi]$



Considering the arcs of the sinusoid corresponding to the intervals $[-\pi, 0]$ and $[0, \pi]$ we readily observe that the graph is concave (or convex down) for $x \in [-\pi, 0]$, that is it lies above the tangent drawn at any point of that interval, while the graph is convex (or convex up) for $x \in [0, \pi]$ that is it lies below the tangent drawn at any point of that interval. However, it should be noted, that the arc of sinusoid corresponding to the whole interval $[-\pi, \pi]$ is neither convex nor concave.

Theorem. (Sufficient condition for the convexity (concavity) of the graph of a function).

If $f''(x) < 0$ on the open interval (a, b) then the graph of the function is convex on that interval.

If $f''(x) > 0$ on the open interval (a, b) then the graph of the function is concave on that interval.

Proof. Let $f''(x) < 0$ at all points $x \in (a,b)$. Let us take an arbitrary point $x_0 \in (a,b)$ and draw the tangent to the graph of the function $y = f(x)$ at the point $(x_0, f(x_0))$. The equation of the tangent has the form

$$y = f(x_0) + f'(x_0)(x - x_0) \equiv T(x)$$

We have to prove that all the points of the graph of the function $y = f(x)$ lie below its every tangent $y = T(x)$. To prove it, we have to show that the ordinate of any point of the graph of the function $y = f(x)$ is less than the ordinate of the tangent $y = T(x)$ at the same value of *x*. To this end, we consider the difference

$$f(x) - T(x) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Applying the Lagrange's theorem to the difference $f(x) - f(x_0)$ we get

$$f(x) - T(x) = f'(c) \cdot (x - x_0) - f'(x_0)(x - x_0) =$$
$$[f'(c) - f'(x_0)] \cdot (x - x_0)$$

where the point *c* lies between x_0 and *x*.

Now applying again the Lagrange's theorem to the difference $f'(c) - f'(x_0)$ we obtain

$$f(x) - T(x) = f''(\xi)(c - x_0) \cdot (x - x_0)$$

and $f''(\xi) < 0$ since $f''(x) < 0$ for $x \in (a,b)$.

There are two possible cases here: $x > x_0$ or $x < x_0$. It is clear that

$$x > x_0 \Rightarrow x_0 < c < x \Rightarrow (c - x_0)(x - x_0) > 0 \Rightarrow f''(\xi)(c - x_0)(x - x_0) < 0 \text{ since } f''(\xi) < 0,$$

$$x < x_0 \Rightarrow x < c < x_0 \Rightarrow (c - x_0)(x - x_0) > 0 \Rightarrow f''(\xi)(c - x_0)(x - x_0) < 0 \text{ since } f''(\xi) < 0.$$

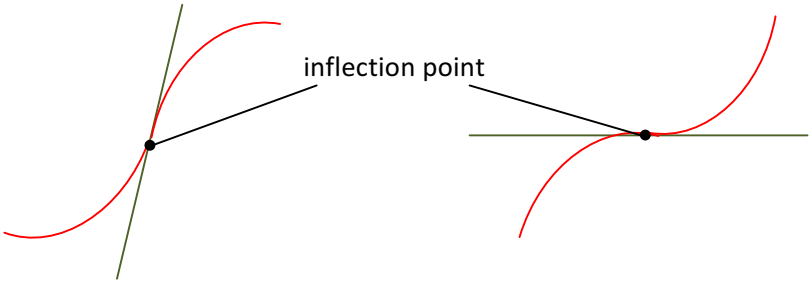
Therefore in any case $f(x) - T(x) < 0$, t.e. $f(x) < T(x)$,which is what we had to prove.

The case when $f''(x) > 0$ is considered similarly.

An important characteristic of a curve are the points separating its convex and concave arcs.

Definition. The point $(x_0, f(x_0))$ of the graph of the function separating its convex part from the concave part is called **a point of inflexion.**

At a point of inflection the tangent intersects the curve. In the vicinity of such a point the curve lies on both sides of its tangent drawn through that point.



This does not require the point to be a stationary point. Geometrically, the portion of the graph around a point of inflexion can interpreted as “S-bend” which is equivalent to the curve switching from convex to concave or vice versa.

These considerations lead to a new definition.

Definition. Let $y = f(x)$ be differentiable at a point x_0 and let $y = T(x)$ be the equation of a tangent to the graph of the function at the point $(x_0, f(x_0))$.

If the difference $f(x) - T(x)$ changes sing as x passes through the point x_0 **is a point of in inflexion.**

Theorem. (The first sufficient condition for the inflexion point) Let $f(x)$ be differentiable at a point x_0 and possesses the second derivative $f''(x)$ in a neighbourhood of x_0 except possible the point x_0 itself.

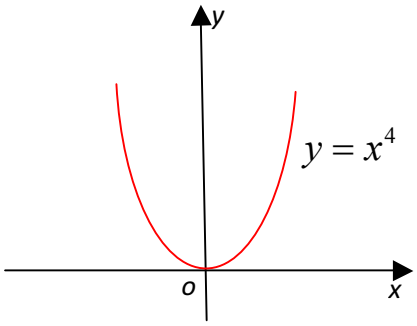
If the second derivative $f''(x)$ changes sign as x passes through the point x_0 then x_0 is a point of inflexion of the function $f(x)$.

Theorem. (The second sufficient condition for the inflexion point) If $f''(x) = 0$ and $f'''(x_0) \neq 0$ then x_0 is a point of inflexion.

Theorem. (Necessary condition for a point of inflexion). If x_0 is the abscissa of the inflexion point of the graph of the function $y = f(x)$, then the second derivative $f''(x_0)$ is equal to zero or does not exist.

Thus, briefly, at a point of inflexion $f''(x_0) = 0$. This proposition provides a necessary condition for the point of inflexion.

Although a point of inflexion must have zero second derivative, $f''(x_0)=0$, the converse is not true, that is by far not every root of the equation $f''(x)=0$is the abscissa of a point of inflexion. For example, we can see this from the function $f(x)=x^4$.



The first and the second derivatives of the function $f(x)=x^4$are $f'(x)=4x^3$and $f''(x)=14x^2$. Although $f''(0)=0$, the point (0,0) is the vertex of the parabola $y=x^4$and does not serve as its point of inflexion.

Examples.

1.Find the intervals of convexity and concavity of the graph of the function $y=x^5+5x-8$.

Solution. We have $y'=5x^4+5$, $y''=20x^3$.

It is clear that

if $x>0$, then $y''>0$and the curve is concave

if $x<0$, then $y''<0$and the curve is convex.

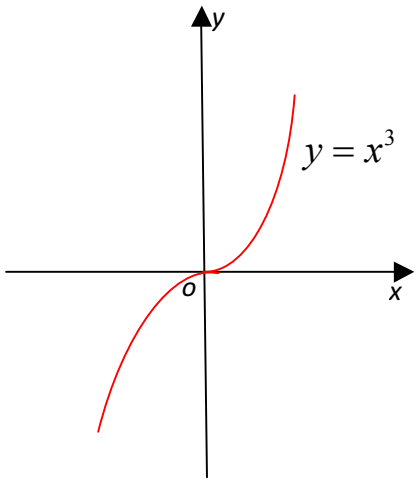
Thus we see that the curve is convex on the interval $(-\infty,0)$ and concave on the interval $(0,\infty)$.

2. Find the coordinates of the point of inflexion on the curve $y=x^3$.

Solution. We have $y' = 3x^2$, $y'' = 6x$. Now we equate the second derivative to zero:

$6x = 0$, i.e. $x = 0$.

Thus, the point (0,0) is an inflexion point.



3. The curve $y = x^4 - 6x^2 + 7x + 2$ has two points of inflexion. Find their coordinates.

Solution. We have $y' = 4x^3 - 12x + 7$, $y'' = 12x^2 - 12$.

At a point of inflexion $y'' = 0$: $12x^2 - 12 = 0$ i.e. $x = \pm 1$.

Thus, the points (-1,-10) and (1,4) are the points of inflexion.

Asymptotes

When studying the behaviour of a function or when investigating the graph of a function as its argument tends to plus or minus infinity we deal with the branches of its graph receding to infinity.

Definition. A straight line L is called an asymptote to the curve $y = f(x)$ if the distance between the point $P(x, y)$ of the curve and the line L tends to zero as the distance from this point to the origin increases indefinitely.

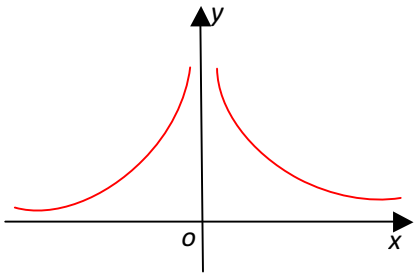
A graph of $y = f(x)$ may have vertical, horizontal and inclined asymptotes.

Definition. A straight line *x*=*a* is called **a vertical asymptote** of the curve $y = f(x)$ if $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$.

There may be any number of vertical asymptotes, even an infinite number.

Examples.

1. $y = \dfrac{1}{x^2}, \qquad \lim\limits_{x \to 0} \dfrac{1}{x^2} = +\infty$

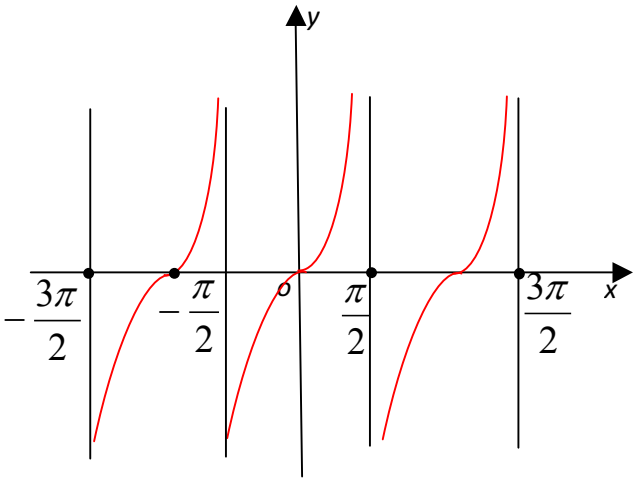


The straight line *x*=0 is a vertical asymptote of the curve $y = \dfrac{1}{x^2}$.

2. $y = \tan x$

This function has an infinite number of asymptotes:

$$y = \frac{\pi}{2} + \pi k, \qquad k \in Z$$

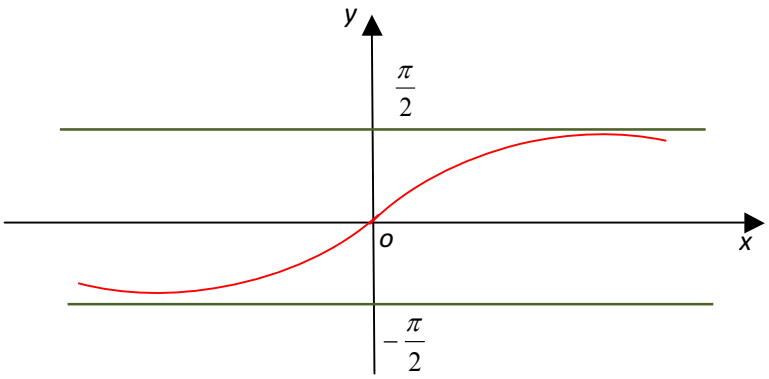


Definition. A straight line *y*=*b* is called **a horizontal asymptote** of the curve $y = f(x)$ if there exist a limit $\lim\limits_{x \to +\infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$.

There cannot be more than two horizontal asymptotes.

Examples.

1. $y = \arctan x$

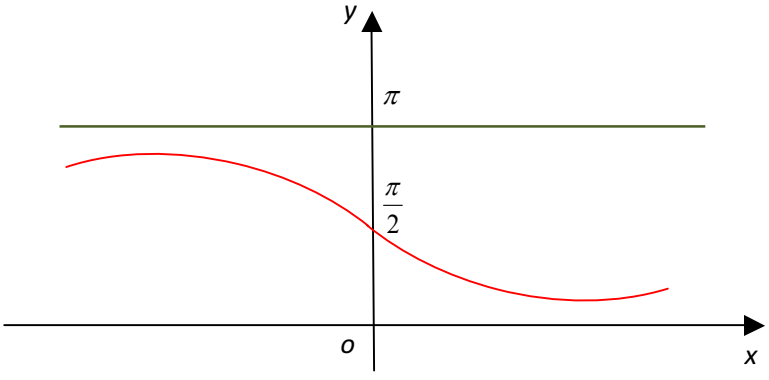


$\lim\limits_{x \to +\infty} \arctan x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2}$ is a horizontal asymptote,

$\lim\limits_{x \to -\infty} \arctan x = -\frac{\pi}{2} \Rightarrow y = -\frac{\pi}{2}$ is a horizontal asymptote.

Thus, this function has two asymptotes: $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

2. $y = arc\cot x$

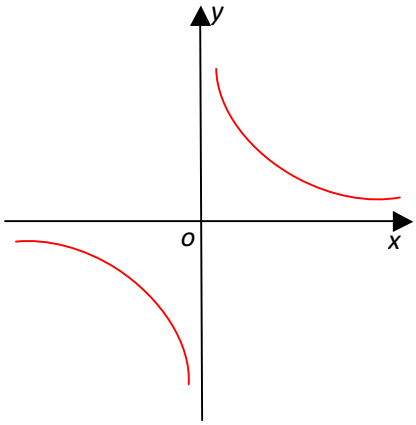


$\lim\limits_{x\to+\infty} arc\cot x = 0 \Rightarrow y = 0$ is a horizontal asymptote,

$\lim\limits_{x\to-\infty} arc\cot x = \pi \Rightarrow y = \pi$ is a horizontal asymptote.

Therefore, this function has two asymptotes:

$y = 0$ and $y = \pi$.



3. $y = \dfrac{1}{x}$

$\lim\limits_{x\to\infty} \dfrac{1}{x} = 0 \Rightarrow y = 0$ is a horizontal asymptote,

$\lim\limits_{x\to+0} \dfrac{1}{x} = +\infty,\ \lim\limits_{x\to-0} \dfrac{1}{x} = -\infty \Rightarrow x = 0$ is a vertical asymptote.

Thus, this function has one vertical and one horizontal asymptotes.

Definition. A straight line $y = kx + b$ is called **an inclined asymptote** of the curve $y = f(x)$ if there exist limits

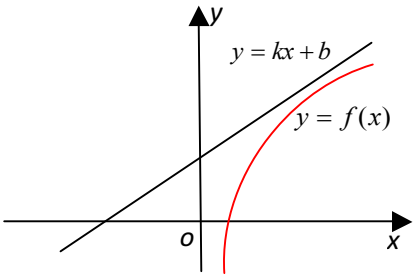
$$k = \lim_{x\to+\infty} \frac{f(x)}{x}, \quad b = \lim_{x\to+\infty} [f(x) - kx]$$

or

$$k = \lim_{x \to -\infty} \frac{f(x)}{x}, \quad b = \lim_{x \to -\infty} [f(x) - kx].$$

There cannot be more than two inclined asymptotes corresponding to $x \to +\infty$ and $x \to$ -∞

These asymptotes are found as follows: Let a straight line $y = kx + b$ be an asymptote of the graph of $y = f(x)$ as $x \to \infty$.



Then the difference *d* between the asymptote $y = kx + b$ and the curve $y = f(x)$ is equal to

$$d = (kx + b) - f(x)$$

and tends to zero as $x \to \infty$.

It follows that

$$f(x) = kx + b - d$$

and, hence

$$\frac{f(x)}{x} = k + \frac{b}{x} - \frac{d}{x} \to k$$

that is

$$k = \lim_{x\to\infty} \frac{f(x)}{x}$$

Besides ,

$$d = (kx + b) - f(x) \Rightarrow f(x) - kx = b - d \to b$$

That is

$$b = \lim_{x\to\infty} [f(x) - kx].$$

Each of these limits must exist and be finite, otherwise there will be no asymptote as $x \to \infty$. If these finite limits do exist then the asymptote also exists since it is seen from the last equality that the value $[f(x) - kx] - b$ tends to zero as $x \to \infty,$ i.e.

$$\lim_{x\to\infty} [f(x) - (kx + b)] = 0$$

Examples.

1. Find the asymptotes of the curve $y = \frac{x^3}{x^2 - 1}$

Solution.

The graph of this function has two vertical asymptotes:

$x = -1$ and $x = 1$, since

$$\lim_{x\to -1} \frac{x^3}{x^2 - 1} = -\infty \text{ and } \lim_{x\to 1} \frac{x^3}{x^2 - 1} = +\infty$$

To determine its inclined asymptote we compute the limits

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^3}{x(x^2-1)}=\lim_{x\to\infty}\frac{x^2}{x^2-1}=1$$

$$b=\lim_{x\to\infty}\left[f(x)-kx\right]=\lim_{x\to\infty}\left[f(x)-x\right]=\lim_{x\to\infty}\left[\frac{x^3}{x^2-1}-x\right]=\lim_{x\to\infty}\frac{x}{x^2-1}=0$$

Since *k*=1 and *b*=0 the asymptote has the form *y*=*x*. Thus, the bisector of the first quadrant *y*=*x* is the only inclined asymptote the graph possesses.

Lecture 12

Indefinite Integral

Antiderivative

When differentiating a function we solve the problem of finding of the derivative of a given function. Now we want to solve the inverse problem: given the derivative of a function, to find this function.

Definition. The function $F(x)$ is called an antiderivative or a primitive of the given function $f(x)$ in a given interval if $F'(x)=f(x)$ or $dF(x)=f(x)dx$.

Let us consider some examples.

1. The function $F'(x)=\frac{x^5}{5}$ is an antiderivative of the function $f(x)=x^4$ in the interval $(-\infty,+\infty)$, since

$$F'(x)=\left(\frac{x^5}{5}\right)'=\frac{1}{5}\cdot\left(x^5\right)'=\frac{1}{5}\cdot 5x^4=x^4=f(x)$$

2. The function $F(x) = \sin^2 x$ is a antiderivative of the function $f(x) = \sin 2x$ in the interval $(-\infty, +\infty)$, since

$$F'(x) = \left(\sin^2 x\right)' = 2 \cdot \sin x \cdot (\sin x)' = 2 \cdot \sin x \cdot \cos x = \sin 2x = f(x)$$

It is easy to see, from these two examples, that the derivatives of $\frac{x^5}{5} + C$ and $\sin^2 x + C$, where C is an arbitrary constant, are also equal respectively to x^4 and $\sin 2x$. These examples indicate that a given function has infinitely many antiderivatives because the constant C can take on arbitrary values.

Thus we see that the problem of finding an antiderivative has infinitely many solutions.

Properties of an antiderivative

1. If the function $f(x)$ possesses an antiderivative $F(x)$, then it possesses infinitely many antiderivatives, all the antiderivatives being contained in the expression $F(x) + C$, where C is an arbitrary constant.

2. Any two primitives differ by a constant.

Proof. Let F_1 and F_2 be two antiderivatives of f. It means that $F_1' = f$ and $F_2' = f$. Let us consider the difference $F_1 - F_2$ and find its derivative. We have

$$(F_1 - F_2)' = F_1' - F_2' = f - f = 0$$

It follows that $F_1 - F_2 = C$, where C is a constant.

3. If the function *f* possesses an antiderivatives *F* and the function *g* possesses an antiderivatives *G*, then the sum $f + g$ possesses an antiderivative $F + G$.

Proof. Indeed since $F' = f$ and $G' = g$ we have

$$(F+G)' = F' + G' = f + g$$

4. If the function *f* possesses an antiderivatives *F*, then the function kf, where *k* is a constant, possesses an antiderivative kF.

Proof. Indeed we have

$$(kF)' = k(F)' = k \cdot f$$

5. If the function $f(x)$ possesses a primitive $F(x)$, then the function $f(t)$ possesses a primitive $F(t)$.

6. If the function $f(x)$ possesses a primitive $F(x)$, then the function $f(ax+b)$ possesses a primitive $\frac{1}{a}F(ax+b)$, where *a* and *b* are constant and also $a \neq 0$.

Proof. According to the rule for finding the derivative of a composite function we have

$$\left(\frac{1}{a}F(ax+b)\right)' = \frac{1}{a}F'(ax+b)\cdot a = f(ax+b).$$

Indefinite Integral and its Properties

Definition. The collection of all antidrivatives of a function $f(x)$ is called the indefinite integral and is denoted by the symbol $\int f(x)dx$.

$$\int f(x)dx = F(x) + C$$

Here $\int$ is the integral sign, $f(x)$ is the integrand, $f(x)dx$ is the element of integration, and *x* is the variable of integration.

The process of finding an indefinite integral is called integration of a function.

To find the indefinite integral of a function means to find all its antiderivatives (for which it is sufficient to determine one of them). This is the reason why we speak of indefinite integration as it is not indicated which of the antiderivatives is meant.

The graph of an antiderivative of a function $f(x)$ is called an integral curve of the function $y = f(x)$. It is clear that any integral curve can be obtained by a translation (parallel displacement) of any other integral curve in the vertical direction. Therefore, geometrically, the indefinite integral is represented by the set (collection) of all integral curves obtained by continuous parallel displacement of one of them in then vertical direction(Fig1.1).

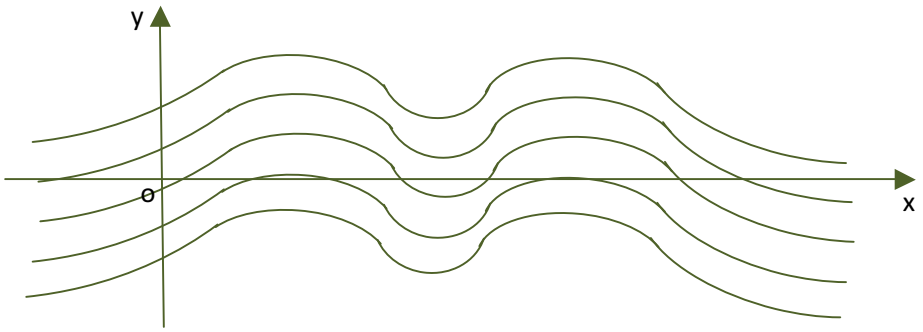


Fig1.1

Properties of an Indefinite Integral (rules of integration)

1. $\int (f(x)dx)' = f(x)$

Proof. By the definition of the indefinite integral we have

$$(\int f(x)dx)' = (F(x)+C)' = F'(x)+C' = F'(x) = f(x)$$

2. $d\int (f(x)dx) = f(x)dx$

Proof. According to the definitions of the differential and the indefinite integral we have

$$d(\int f(x)dx) = (\int f(x)dx)' == f(x)dx$$

3. $\int f'(x)dx = f(x)+C$ or $\int df(x) = f(x)+C$

Thus, the signs of integration and differentiation mutually cancel out. The result of computing an indefinite integral can always be verified by finding the derivative of the result.

4. A constant factor in the integrand can be taken outside the sign of the integral:

$\int Cf(x)dx = C\int f(x)dx$, C=constant

5. The integral of a sum of a finite number of functions is equal to the sum of the integrals of these functions:

$$\int [f_1(x)+f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx,$$

$$\int [f_1+f_2+\ldots+f_n]dx = \int f_1 dx + \int f_2 dx + \ldots + \int f_n dx$$

6. On the invariance of integration formulas.

If $\int f(x)dx = F(x) + C$ and $u = \varphi(x)$ any function possessing a continuous derivative, then $\int f(u)du = F(u) + C$.

Proof. From $\int f(x)dx = F(x) + C$ it follows that $F'(x) = f(x)$.

For the composite function $F(u) = F(\varphi(x))$ its differential is

$dF(u) = F'(u)du = f(u)du$.

Therefore $\int f(u)du = \int dF(u) = F(u) + C$

Table of Basic Integrals

1. $\int dx = x + C$

2. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$ for $\alpha \neq -1$

3. $\int \frac{1}{x} dx = \ln|x| + C$

4. $\int \sin x dx = -\cos x + C$

5. $\int \cos x dx = \sin x + C$

6. $\int \frac{1}{\cos^2 x} dx = \tan x + C$

10. $\int e^x dx = e^x + C$

11. $\int a^x dx = \frac{a^x}{\ln a} + C$

12. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

14. $\int \frac{dx}{1+x^2} = \arctan x + C$

15. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

7. $\int \frac{1}{\sin^2 x} dx = -\cot x + C$

8. $\int \tan x dx = -\ln|\cos x| + C$

9. $\int \cot x dx = \ln|\sin x| + C$

16. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

17. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

Basic Methods of Integration

The basic methods of integration are the following methods:

1. The decomposition method

2. Integration by parts.

3. Integration by change of variable (by substitution).

The Decomposition Method

A given integral can often be represented in the form of a sum of tabular integrals (i.e. the integrals from the table). Then we perform termwise integration and thus obtain the answer. Let us consider several examples.

1. $\int \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n\right) dx = a_0 \int dx + a_1 \int x dx + a_2 \int x^2 dx + \ldots + a_n \int x^n dx =$
$= a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \ldots + \frac{1}{n+1} a_n x^{n+1} + C$

2. $\int \left(3x^2 - 2\sin x + 5e^x\right) dx = \int 3x^2 dx - \int 2\sin x dx + \int 5e^x dx =$

$$3\int x^2 dx - 2\int \sin x dx + 5\int e^x dx = 3\cdot\frac{x^3}{3} - 2\cdot(-\cos x) + 5e^x + C =$$
$$= x^3 + 2\cos x + 5e^x + C$$

3.
$$\int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx =$$
$$= \int\left(\frac{1}{\cos^2 x} - 1\right) dx = \int \frac{1}{\cos^2 x} dx - \int dx = \tan x - x + C$$

Integration by parts

The method of integration by parts is implied by formula for differentiating a product of two functions. Let $u = u(x)$ and $v = v(x)$ be functions of *x* possessing continuous derivatives. We have

$$d(uv) = udv + vdu$$

whence

$$udv = d(uv) - vdu$$

Integrating both sides of the latter equality, we get

$$\int udv = \int d(uv) - \int vdu$$

that is,

$$\int u\,dv = uv - \int v\,d\,u \qquad (*)$$

This is the formula of integration by parts which can be rewritten as follows

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

We do not write down the arbitrary constant appearing in the integration of $d(uv)$ and include it into the arbitrary constant in the second integral on the right-hand side of equality.

Let us consider some examples.

1. Let us find $\int xe^x dx$

We put $x = u, \quad e^x dx = dv,$ whence

$du = dx$, and $v = \int e^x dx = e^x$.

Now using the formula of integration by parts (*) we get

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

2.Let us find $\int x\cos x dx$.

Here we put $x = u$, $\cos x dx = dv$, whence

$du = dx$ and $v = \int \cos x dx = \sin x$

By formula (*) we obtain

$$\int x\cos x dx = x \cdot \sin x - \int \sin x dx = x \cdot \sin x + \cos x + C$$

3. Take the integral $\int x^2 \cos x dx$.

Here we put $x^2 = u, \ \cos x dx = dv,$ whence

$du = 2xdx$ and $v = \sin x$.

According to the formula (*) we have

$$\int x^2 \cos x dx = x^2 \cdot \sin x - 2\int x \sin x dx.$$

The last integral is again found with the aid of integration by parts. Finally we obtain

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2\sin x + C.$$

4. Let us find the integral $\int x \ln x dx$.

We put $\ln x = u, \quad xdx = dv$

Whence

$$du = \frac{1}{x}dx, \quad \text{and } v = \int xdx = \frac{x^2}{2}.$$

Then by the formula (*) we get

$$\int x \ln x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int xdx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

5. Now let us consider the integral $\int x^n \ln x dx \ \ (n \neq -1)$

Putting $\ln x = u, \ x^n dx = dv$ we find

$$du = \frac{1}{x}dx \text{ and } v = \int x^n dx = \frac{x^{n+1}}{n+1}$$

and thus we have

$$\int x^n \ln x dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1} d\ln x = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

6. Consider the integral $\int e^x \cos x dx$.

We put $\cos x = u, \quad e^x dx = dv$ whence

$du = -\sin x dx$ and $v = \int e^x dx = e^x$.

By formula (*) we obtain

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

Let us again apply integration by parts. On putting $\sin x = u, \quad e^x dx = dv$ we find $du = \cos x dx$, $v = e^x$ and hence

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

Now substituting the expression thus obtained into the result of the first integration we get

$$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

On transposing the integral from the right-hand side to the left we find

$$\int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x) + C$$

Integration by change of variable

A change of variable in an indefinite integral is performed by means of substitutions of two types:

I. $x = \varphi(t)$, where $\varphi(t)$ is a monotonic, continuously differentiable function of the new variable *t*. In this case, the integration is carried out by the formula

$$f(x)dx = \int f[\varphi(t)] \cdot \varphi'(t)dt;$$

II. $u = \psi(x)$, where *u* is a new variable. With the aid of this substitution, the integration is performed by the formula.

$$f(x)dx = \int f[\psi(x)] \cdot \psi'(x)dx = \int f(u)du$$

Let us consider several examples.

1. Find the integral $\int (5x+2)^8 dx$

Solution. Putting $5x+2=t$, we get $5dx = dt$, i..e. $dx = \frac{1}{5}dt$, which yields

$$\int (5x+2)^8 dx = \frac{1}{5}\int t^8 dt = \frac{1}{5}\cdot\frac{1}{9}t^9 + C = \frac{1}{45}(5x+2)^9 + C.$$

2. Find the integral $\int \cos(4x+1)dx$.

Solution. We put $4x+1=t$. Whence $4dx = dt$, i.e. $dx = \frac{1}{4}dt$.

Therefore

$$\int \cos(4x+1)dx = \frac{1}{4}\int \cos t dt = \frac{1}{4}\sin t + C = \frac{1}{4}\sin(4x+1) + C$$

3. Find the integral $\int \sin^3 x \cos x dx$.

Solution. Putting $\sin x = u$ we get $du = d(\sin x) = \cos x dx$ and hence

$$\int \sin^3 x \cos x dx = \int u^3 du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4 x + C.$$

4. Find the integral $\int e^{\sin x} \cos x dx$.

Solution. Putting $\sin x = u$ we obtain $du = \cos x dx$ and therefore

$$\int e^{\sin x} \cos x dx = \int e^u du = e^u + C = e^{\sin x} + C.$$

5. Find the integral $\int \frac{\arctan x}{1+x^2} dx$.

Solution. Putting $\arctan x = u$ we get $du = \frac{1}{1+x^2} dx$ and hence

$$\int \frac{\arctan x}{1+x^2} dx = \int u du = \frac{u^2}{2} + C = \frac{1}{2}\arctan^2 x + C$$

6. Find the integral $\int x^2 \sqrt[3]{4-3x^3} dx$

Solution. We put $4 - 3x^3 = u$ whence $-9x^2 dx = du$ i.e. $x^2 dx = -\frac{1}{9} du$. Thus,

$$\int x^2 \sqrt[3]{4-3x^3} dx = \int \sqrt[3]{4-3x^3} x^2 dx = -\frac{1}{9}\int \sqrt[3]{u} du = -\frac{1}{9}\int u^{\frac{1}{3}} du =$$

$$= -\frac{1}{9}\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = -\frac{1}{12}u^{\frac{4}{3}} + C = -\frac{1}{12}(4-3x^3)\sqrt[3]{4-3x^3} + C.$$

Recurrence Formula for Finding the Integral

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

The desired formula is obtained by means of integration by parts. We have

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{(x^2+a^2)^n}dx = \frac{1}{a^2}\int \frac{a^2+x^2}{(x^2+a^2)^n}dx -$$
$$-\frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^n}dx = \frac{1}{a^2}I_{n-1} - \frac{1}{a^2}\int x\cdot\frac{x}{(x^2+a^2)^n}dx.$$

Let us put $u = x,\ dv = \frac{x}{(x^2+a^2)^n}dx$; then $du = dx$ and

$$v = \int \frac{xdx}{(x^2+a^2)^n} = \frac{1}{2}\int (x^2+a^2)^{-n}d(x^2+a^2) = -\frac{1}{2(n-1)}\cdot\frac{1}{(x^2+a^2)^{n-1}}$$

Consequently

$$I_n = \frac{1}{a^2}I_{n-1} + \frac{1}{a^2}\left[\frac{x}{2(n-1)(x^2+a^2)^{n-1}} - \frac{1}{2(n-1)}\int\frac{dx}{(x^2+a^2)^{n-1}}\right],$$

or

$$I_n = \frac{1}{a^2}I_{n-1} + \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} - \frac{1}{2a^2(n-1)}I_{n-1}$$

that is

$$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{1}{a^2}\frac{2n-3}{2n-2}I_{n-1}.$$

For $n=1$ we have $I_1 = \int\frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$.

Putting $n=2$, we obtain an expression for the integral I_2 in terms of the elementary functions:

$$I_2 = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot I_1 = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\left(\frac{1}{a}\arctan\frac{x}{a} + C\right).$$

Assuming now $n = 3$, we find the integral I_3 (the integral I_2 having been found). In this way we find I_n for any positive integer n.

Integration of Rational Functions (Rational Fractions)

Definition. A rational function (rational fraction) is a fraction of the from

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

Definition. A rational fraction is said to be **proper** if the degree of the numerator $P(x)$ is less than the degree of the denominator; otherwise the fraction is said to be **improper.**

Let us consider the integral of the form

$$\int \frac{P(x)}{Q(x)} dx$$

If $\frac{P(x)}{Q(x)}$ is an improper, then it can be represented in the form of a sum of a polynomial and a proper fraction:

$$\frac{P(x)}{Q(x)} = L(x) + \frac{M(x)}{Q(x)}$$

Consequently.

$$\int \frac{P(x)}{Q(x)} dx = \int L(x) dx + \int \frac{M(x)}{Q(x)} dx$$

The integration of the polynomial $L(x) = a_0 + a_1 x + \ldots + a_n x^n$ does not present any difficulties, and hence the problem reduces to integrating the proper fraction in which the degree of the numerator $M(x)$ is less that of the denominator $Q(x)$. The term partial (elementary) fractions is used for proper fractions of the following types:

I. $\frac{A}{x-a}$;

II. $\frac{A}{(x-a)^n}$, where *n* is a positive integer;

III. $\frac{Ax+B}{x^2+px+q}$, where $\frac{p^2}{4} - q < 0$, that is the quadratic trinomial $x^2 + px + q$ does not posses real roots;

IV. $\frac{Ax+B}{(x^2+px+q)^n}$, where *n* is a positive integer, and $\frac{p^2}{4} - q < 0$.

In all the four cases it is assumed that A, B, p, q, a are real numbers. The fractions we have enumerated will be called, respectively, partial fractions of types I, II, III and IV.

It turns out that the proper rational fraction $\frac{M(x)}{Q(x)}$ can be represented as a decomposition into partial fractions.

Thus we need to know how to find the integrals of the partial fractions.

Let us consider the integrals of the partial fractions of the first three types. We have

I. $\int \frac{A}{x-a}dx = A \cdot \ln|x-a| + C$

Indeed, putting $x - a = t$ we find $dx = dt$ and hence

$$\int \frac{A}{x-a}dx = \int \frac{A}{t}dt = A\int \frac{1}{t}dt = A\ln|t| + C = A\ln|x-a| + C$$

II. $\int \frac{A}{(x-a)^n}dx = -\frac{A}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + C$

Here we do the same: $x - a = t \Rightarrow dx = dt$ and we get

$$\int \frac{A}{(x-a)^n}dx = \int \frac{A}{t^n}dt = A\int t^{-n}dt = -\frac{A}{n-1} \cdot \frac{1}{t^{n-1}} + C = -\frac{A}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + C$$

III. $\int \frac{dx}{x^2 + px + q} = \frac{2}{\sqrt{4q - p^2}} \arctan \frac{2x + p}{\sqrt{4q - p^2}} + C$

Indeed, for this special case of the partial fraction of type III we obtain

$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}$ or $x^2 + px + q = t^2 + a^2$,

where $t = x + \frac{p}{2}$, $a = \frac{\sqrt{4q - p^2}}{2}$ (here $\frac{p^2}{4} - q < 0$), whence

$$\int \frac{dx}{x^2 + px + q} = \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan \frac{t}{a} + C = \frac{2}{\sqrt{4q - p^2}} \arctan \frac{2x + p}{\sqrt{4q - p^2}}$$

Example. Find the integral $\int \frac{dx}{x^2 + 8x + 32}$

Solution, we have

$$\int \frac{dx}{x^2+8x+32} = \int \frac{dx}{(x+4)^2+16} = \int \frac{d(x+4)}{(x+4)^2+16} = \frac{1}{4}\arctan\frac{x+4}{4} + C$$

We will show now how to integrate, in a general form, partial fractions of type III.

It is required to find $\int \frac{Ax+B}{x^2+px+q}dx$, where $\frac{p^2}{4} - q < 0$. Let us isolate the derivative of the denominator from the numerator of the fraction. To do that, we will represent the numerator in the form

$$Ax + B = \frac{A}{2}(2x+p) + B - \frac{Ap}{2}$$

Then

$$\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{\frac{A}{2}(2x+p)+B-\frac{Ap}{2}}{x^2+px+q}dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \left(B - \frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q}$$

In the first integral the numerator is the derivative of the denominator; therefore,

$$\int \frac{2x+p}{x^2+px+q}dx = \int \frac{d(x^2+px+q)}{x^2+px+q} = \int \frac{du}{u} = \ln|u| + C = \ln(x^2+px+q) + C$$ (here $x^2+px+q > 0$)

The second integral, as has been indicated, can be found by the formula

$$\int \frac{dx}{x^2+px+q} = \frac{2}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$$

Thus, finally we have

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln(x^2+px+q) + \frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$$

Example. Find the integral $\int \frac{(3x-1)dx}{4x^2-4x+17}$

Solution. We have

$$\int \frac{(3x-1)dx}{4x^2-4x+17} = \int \frac{\frac{3}{8}(8x-4)-1+\frac{3}{2}}{4x^2-4x+17}dx = \frac{3}{8}\int \frac{8x-4}{4x^2-4x+17}dx + \frac{1}{2}\int \frac{dx}{4x^2-4x+17} =$$

$$\frac{3}{8}\int \frac{d(4x^2-4x+17)}{4x^2-4x+17} + \frac{1}{2}\cdot\frac{1}{4}\int \frac{dx}{x^2-x+\frac{17}{4}} = \frac{3}{8}\int \frac{du}{u} + \frac{1}{8}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+4} = \frac{3}{8}\ln|u| + \frac{1}{8}\cdot\int \frac{dt}{t^2+2^2} =$$

$$= \frac{3}{8}\ln(4x^2-4x+17) + \frac{1}{8}\cdot\frac{1}{2}\cdot\arctan\frac{t}{2} + C = \frac{3}{8}\ln(4x^2-4x+17) + \frac{1}{16}\cdot\arctan\frac{2x-1}{4} + C\ .$$

Now let us show how to integrate the partial fractions of type IV. It is required to find the integral

$\int \frac{Ax+B}{(x^2+px+q)^n}dx$, where $\frac{p^2}{4}-q<0$.

Let us isolate the derivative of the denominator from the numerator of the fraction:

$$\int \frac{Ax+B}{(x^2+px+q)^n}dx = \int \frac{\frac{A}{2}(2x+p)+B-\frac{Ap}{2}}{(x^2+px+q)^n}dx = \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^n}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^n}$$

The first integral on the right-hand side is easily found with the aid of the substitution $x^2+px+q=t$ whence $(2x+p)dx=dt$. We have

$$\int \frac{2x+p}{(x^2+px+q)^n}dx = \int \frac{dt}{t^n} = -\frac{1}{(n-1)t^{n-1}} + C = -\frac{1}{(n-1)(x^2+px+q)^{n-1}} + C$$

The second integral on the right-hand side must be transformed as follows:

$$\int \frac{2x+p}{(x^2+px+q)^n} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p}{4}\right)^2\right]^n}$$

Putting now $x+\frac{p}{2}=t,\ dx=dt$ and introducing the designation $q-\frac{p^2}{4}=a^2$ we get

$$\int \frac{dx}{(x^2+px+q)^n} = \int \frac{dt}{\left(t^2+a^2\right)^n}$$

Thus we see that integration of an elementary fraction of type IV can be carried out with the aid of the recurrence formula.

Thus the integral of every rational fraction (rational function) can be reduced to integrals of partial fractions of the indicated types. Besides, the integral of a rational fraction is always expressible in terms of elementary functions such as rational functions, the logarithmic function and the arctangent.

Example. Find the integral $\int \frac{2x+3}{(x^2+2x+5)^2}dx$.

Solution. We have

$$\int \frac{2x+3}{(x^2+2x+5)^2}dx = \int \frac{2x+2+1}{(x^2+2x+5)^2}dx = \int \frac{2x+2}{(x^2+2x+5)^2}dx + \int \frac{1}{[(x+1)^2+4]^2}dx.$$

We perform the change of variable $x^2+2x+5=t,\ (2x+2)dx=dt$ in the first integral and put $x+1=z,\ dx=dz$ in the second integral. We get

$$\int \frac{2x+3}{(x^2+2x+5)^2}dx = \int \frac{dt}{t^2} + \int \frac{dz}{(z^2+4)^2} = -\frac{1}{t} + \frac{1}{2\cdot 4}\frac{z}{z^2+4} + \frac{1}{2\cdot 4}\left(\frac{1}{2}\arctan\frac{z}{2}+C\right)$$

Returning now to the old variable, we get

$$\int \frac{2x+3}{(x^2+2x+5)^2}dx = -\frac{1}{x^2+2x+5} + \frac{1}{8}\cdot\frac{x+1}{x^2+2x+5} + \frac{1}{16}\arctan\frac{x+1}{2} + C =$$
$$= \frac{x-7}{8(x^2+2x+5)} + \frac{1}{16}\arctan\frac{x+1}{2} + C$$

In the conclusion we formulate the main stages of integration the rational fractions.

Before integrating the rational fraction $\frac{P(x)}{Q(x)}$, we must perform the following algebraic transformations and calculations:

I. If we are given an improper rational fraction, we have to isolate its entire rational function (a polynomial), that is, represent in the form

$$\frac{P(x)}{Q(x)} = L(x) + \frac{M(x)}{Q(x)}$$

Where $L(x)$ is a polynomial and $\frac{M(x)}{Q(x)}$ is a proper rational fraction.

II. Factor the denominator of the fraction into linear and quadratic multipliers:

$Q(x) = (x-a)^n \ldots (x^2+px+q)^k \ldots$, where $\frac{p^2}{4} - q < 0$.

III. Decompose the proper rational fraction into partial fractions:

$$\frac{M(x)}{Q(x)} = \frac{A_1}{(x-a)^n} + \frac{A_1}{(x-a)^{n-1}} + \ldots + \frac{A_n}{x-a} + \ldots$$
$$\ldots + \frac{B_1x+C_1}{(x^2+px+q)^k} + \frac{B_2x+C_2}{(x^2+px+q)^{k-1}} + \ldots + \frac{B_kx+C_k}{x^2+px+q} + \ldots$$

IV. Compute the undetermined coefficients $A_1, A_2, \ldots, A_n, B_1, C_1, B_2, C_2, \ldots B_k, C_k, \ldots$ for which purpose it is necessary to reduce the latter equality to the common denominator, equate the coefficients in the same degrees of *x* on the right-hand and left-hand sides of the identity obtained and solve the system of linear equations with respect to the sought-for coefficients.

Thus, as a result, integration of a rational fraction will reduce to finding integrals of a polynomial and of partial rational fractions.

Let us consider several examples.

1. Find the integral $\int \frac{x-3}{x^3-x}dx$.

Solution. The given proper rational fraction can be represented as a sum of partial fractions of type I:

$$\frac{x-3}{x^3-x} = \frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

First of all we must find the unknown coefficients A, B and C. Clearing the equation of fractions, we obtain.

$$x-3 = A(x^2-1) + Bx(x+1) + Cx(x-1).$$

Consequently, combining the terms of like degrees, we get

$$x-3 = (A+B+C)x^2 + (B-C)x - A.$$

Since it is an identity, the coefficients in like powers of *x* on both sides must be equal:

$$x-3=0\cdot x^2-x-3=(A+B+C)x^2+(B-C)x-A$$

Comparing the coefficients in like powers of *x*, we get the system of equations

$$\begin{cases} A+B+C=0 \\ B-C=1 \\ -A=-3 \end{cases}$$

From which we find $A=3, B=-1$ and $C=-2$.

There is a still simpler method of determined the coefficients A, B and C; putting in succession $x=0, x=1, x=-1$, we obtain $-3=-A,\ -2=2B,$ and $-4=2C,$.that is, again $A=3, B=-1$and $C=-2$.

Thus, decomposition of a rational fraction into partial fractions has the form

$$\frac{x-3}{x^3-x}=\frac{3}{x}-\frac{1}{x-1}-\frac{2}{x+1}$$

and therefore

$$\int\frac{x-3}{x^3-x}dx=3\int\frac{dx}{x}-\int\frac{dx}{x-1}-2\int\frac{dx}{x+1}=3\cdot\ln|x|-\ln|x-1|-2\ln|x+1|+C=\ln\left|\frac{x^3}{(x-1)(x+1)^2}\right|+C$$

2. Find the integral $\int\frac{x^2+1}{(x-1)^3(x+3)}dx$.

Solution. The factor $(x-1)^3$is associated with the sum of three partial fractions $\frac{1}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}$, and the factor $x+3$, with the partial fraction $\frac{D}{x+3}$.

Therefore $\frac{x^2+1}{(x-1)^3(x+3)}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{x+3}$

Clearing the equation of fractions, we get

$$x^2+1=A(x+3)+B(x-1)(x+3)+C(x-1)^2(x+3)+D(x-1)^3.$$

Consequently,

$$x^2+1=A(x+3)+B(x^2+2x-3)+C(x^3+x^2-5x+3)+D(x^3-3x^2+3x-1)$$

Combining the terms of like degrees, we get

$$x^2+1=(C+D)x^3+(B+C-3D)x^2+(A+2B-5C+3D)x+3A-3B+3C-D$$

Comparing the coefficients in like powers of *x*, we get the system of equations

$$\begin{cases} C+D=0 \\ B+C-3D=1 \\ A+2B-5C+3D=0 \\ 3A-3B+3C-D=1 \end{cases} \Rightarrow \begin{cases} C=-D \\ B=4D+1 \\ A=-16D-2 \\ D=-\dfrac{5}{32} \end{cases}$$

from which we find $A=\frac{1}{2},\ B=\frac{3}{8},\ C=\frac{5}{32},\ D=-\frac{5}{32}$.

The unknowns A,B,C and D in the decomposition could have been determined in another way. The real roots of the denominator are the numbers 1 and -3. Putting *x*=1 we get 2=4*A*, i.e. *A*=$\frac{1}{2}$. For $x=-3$, we have $10=-64D$, i.e. $D=-\frac{5}{32}$. Whence $C=\frac{5}{32}$ and $B=\frac{3}{8}$.

Thus, finally decomposition of a rational fraction into partial fractions has the form

$$\frac{x^2+1}{(x-1)^3(x+3)} = \frac{1}{2(x-1)^3} + \frac{3}{8(x-1)^2} + \frac{5}{32(x-1)} - \frac{5}{32(x+3)}$$

Thus, we get

$$\int \frac{x^2+1}{(x-1)^3(x+3)}dx = \frac{1}{2}\int \frac{dx}{(x-1)^3} + \frac{3}{8}\int \frac{dx}{(x-1)^2} + \frac{5}{32}\int \frac{dx}{x-1} - \frac{5}{32}\int \frac{dx}{x+3} =$$
$$= -\frac{1}{4(x-1)^2} - \frac{3}{8(x-1)} + \frac{5}{32}\ln\left|\frac{x-1}{x+3}\right| + C$$

3. Find the integral $\int \frac{x^3-2x}{(x^2+1)^2}dx$

Solution. The given proper rational fraction can be represented as a sum of partial fractions of types III and IV:

$$\frac{x^3-2x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$$

Clearing the equation of fractions, we obtain

$$x^3 - 2x = Ax + B + (Cx+D)(x^2+1).$$

or

$$x^3 - 2x = Ax + B + Cx^3 + Dx^2 + Cx + D.$$

Combining the terms of like degrees, we get

$$x^3 - 2x = Cx^3 + Dx^2 + (A+C)x + B + D.$$

Comparing the coefficients in like powers of *x*, we get the system of equations

$$\begin{cases} C = 1 \\ D = 0 \\ A + C = -2 \\ B + D = 0 \end{cases}$$

from which we find $A = -3,\ B = 0,\ C = 1,\ D = 0$.

Consequently, decomposition of a rational fraction into partial fraction has the form

$$\frac{x^3 - 2x}{(x^2+1)^2} = \frac{-3x}{(x^2+1)^2} + \frac{x}{x^2+1}$$

Therefore, we get

$$\int \frac{x^3 - 2x}{(x^2+1)^2} dx = \int \frac{-3x dx}{(x^2+1)^2} + \int \frac{x dx}{x^2+1} = -\frac{3}{2}\int \frac{d(x^2+1)}{(x^2+1)^2} +$$
$$+ \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{3}{2(x^2+1)} + \frac{1}{2}\ln(x^2+1) + C.$$

4. Find the integral $\int \frac{x^3 + 3x^2 + 5x + 7}{x^2 + 2} dx$.

Solution. Since the integrand is an improper rational fraction, first of all we have to isolate its entire rational function i.e. a polynomial:

$$\begin{array}{l|l} x^3 + 3x^2 + 5x + 7 & x^2 + 2 \\ \hline -\ \cancel{x^3 \quad + 2x} & x + 3 \\ \hline \quad -\ 3x^2 + 3x + 7 & \\ \hline \quad 3x^2 \quad + 6 & \end{array}$$

$3x+1$

Thus, we have

$$\frac{x^3+3x^2+5x+7}{x^2+2}=x+3+\frac{3x+1}{x^2+2}$$

It follows that

$$\int\frac{x^3+3x^2+5x+7}{x^2+2}dx=\int\left(x+3+\frac{3x+1}{x^2+2}\right)dx=\int xdx+3\int dx+\int\frac{3x+1}{x^2+2}dx=$$
$$=\frac{1}{2}x^2+3x+3\int\frac{xdx}{x^2+2}+\int\frac{dx}{x^2+2}=\frac{1}{2}x^2+3x+\frac{3}{2}\int\frac{d(x^2+2)}{x^2+2}+\int\frac{dx}{x^2+2}=$$
$$=\frac{1}{2}x^2+3x+\frac{3}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C\ .$$

Integration of Trigonometric Functions

I. Integrals of the form $\int R(\sin x,\cos x)dx$, where R is a rational function.

Integrals of the indicated form can be reduced to integrals of rational functions with the aid of the so-called **universal trigonometric substitution** $t=\tan\frac{x}{2}$.

As a result of this substitution we have

$$\sin x=\frac{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2};$$

$$\cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2};$$

$\tan\frac{x}{2}=t\Rightarrow\frac{x}{2}=\arctan t\Rightarrow x=2\arctan t$;

$$dx=\frac{2dt}{1+t^2}$$

Thus, we get

$$\int R(\sin xç\cos x)dx=\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2}=\int R_1(t)dt$$

where $R_1(t)$is a rational function.

Example. Find the integral $\int\frac{1}{1+\sin x}dx$.

Solution. The integrand depends rationally on $\sin x$. Using the substitution $t=\tan\frac{x}{2}$, we get

$$\int\frac{1}{1+\sin x}dx=\int\frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}=2\int\frac{dt}{(1+t)^2}=-\frac{2}{1+t}+C=-\frac{2}{1+\tan\frac{x}{2}}+C$$

Example. Find the integral $\int\frac{dx}{\sin x}$.

Solution. We have

$$\int\frac{dx}{\sin x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}=\int\frac{dt}{t}=\ln|t|+C=\ln\left|\tan\frac{x}{2}\right|+C$$

Example. Find the integral $\int\frac{dx}{3\sin x+4\cos x+5}$.

Solution. Here the integrand depends rationally on $\sin x$ and $\cos x$. Therefore, we apply the substitution $t = \tan\frac{x}{2}$; then

$$\int \frac{dx}{3\sin x + 4\cos x + 5} = \int \frac{\frac{2dt}{1+t^2}}{3\cdot\frac{2t}{1+t^2} + 4\cdot\frac{1-t^2}{1+t^2} + 5} = 2\int \frac{dt}{t^2+6t+9} = 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} + C$$

Returning to the old variable, we get

$$\int \frac{dx}{3\sin x + 4\cos x + 5} = -\frac{2}{\tan\frac{x}{2} + 3} + C$$

The universal substitution $t = \tan\frac{x}{2}$ often leads to very complicated expressions containing rational fractions. In certain particular cases it is better to use some other substitutions such as $t = \sin x$, $t = \cos x$ or $t = \tan x$, which we are going to consider here.

I. If $R(\sin x, \cos x)$ is an even function with respect to $\sin x$ and $\cos x$, that is, if $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, then the integral can be rationalized by means of the substitution $t = \tan x$.

Example. Find the integral $\int \frac{1}{\cos^4 x} dx$.

Solution. Using the substitution $t = \tan x$, we get

$$\int \frac{1}{\cos^4 x} dx = \int \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx == \int \left(\tan^2 x + 1\right) d\tan x =$$
$$= \int (t^2 + 1) dt = \frac{t^3}{3} + t + C = \frac{1}{3}\tan^2 x + \tan x + C.$$

Example. Find the integral $\int \frac{\sin^2 x}{\cos^6 x} dx$.

Solution. We have

$$\int \frac{\sin^2 x}{\cos^6 x} dx = \int \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx = \int \tan^2 x \cdot (1 + \tan^2 x) \cdot d \tan x =$$

$$= \int t^2 (1 + t^2) dt = \int (t^2 + t^4) dt = \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C .$$

Example. Find the integral $\int \frac{dx}{\sin^2 x + 2 \sin x \cdot \cos x - \cos^2 x}$.

Solution. The integrand is even with respect to $\sin x$ and $\cos x$. Setting $t = \tan x$, we obtain

$$\sin x = \tan x \cdot \cos x = \frac{\tan x}{\sqrt{\frac{1}{\cos^2 x}}} = \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \frac{t}{\sqrt{1 + t^2}};$$

$$\cos x = \frac{1}{\sqrt{1 + \tan^2 x}} = \frac{t}{\sqrt{1 + t^2}};$$

$$x = \arctan t; \qquad dx = \frac{dt}{1 + t^2} .$$

If follows that

$$\int \frac{dx}{\sin^2 x + 2\sin x \cdot \cos x - \cos^2 x} = \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} + \frac{2t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} - \frac{1}{1+t^2}} =$$

$$= \int \frac{dt}{t^2 + 2t - 1} = \int \frac{d(t+1)}{(t+1)^2 - (\sqrt{2})^2} = \frac{1}{2\sqrt{2}} \ln \left| \frac{t+1-\sqrt{2}}{t+1+\sqrt{2}} \right| + C = \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x + 1 - \sqrt{2}}{\tan x + 1 + \sqrt{2}} \right| + C$$

II. If $R(\sin x, \cos x)$ is an odd function with respect to $\sin x$, that is, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$, the integral can be rationalized by means of the substitution $t = \cos x$.

Example. Find the integral $\int \frac{(\sin x + \sin^3 x)dx}{\cos 2x}$

Solution. Since the integrand is odd with respect to the $\sin x$, we put $t = \cos x$. If follows that $\sin^2 x = 1 - \cos^2 x = 1 - t^2$, $\cos 2x = 2\cos^2 x - 1 = 2t^2 - 1$, $dt = d\cos x = -\sin x dx$.

Thus we have

$$\int \frac{(\sin x + \sin^3 x)dx}{\cos 2x} = \int \frac{(1 + \sin^2 x)\sin x dx}{\cos 2x} = \int \frac{(2 - t^2)(-dt)}{2t^2 - 1} =$$

$$= \int \frac{t^2 - 2}{2t^2 - 1} dt = \frac{1}{2} \int \frac{2t^2 - 4}{2t^2 - 1} = \frac{1}{2} \int \frac{2t^2 - 1 - 3}{2t^2 - 1} dt = \frac{1}{2} \int dt - \frac{3}{2} \int \frac{dt}{2t^2 - 1} =$$

$$= \frac{t}{2} - \frac{3}{2\sqrt{2}} \int \frac{d(\sqrt{2}t)}{(\sqrt{2}t)^2 - 1} = \frac{t}{2} - \frac{3}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}t - 1}{\sqrt{2}t + 1} \right| + C = \frac{1}{2}\cos x - \frac{3}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1} \right| + C$$

III. If $R(\sin x, \cos x)$ is an odd function with respect to $\cos x$, that is, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then the integral can be rationalized by means of the substitution $t = \sin x$.

Example. Find the integral $\int \frac{(\cos^3 x + \cos^5 x)dx}{\sin^2 x + \sin^4 x}$

Solution. Here the integrand is odd with respect to the $\cos x$. Therefore, we use the substitution $t = \sin x$. Then

$\cos^2 x = 1 - \sin^2 x = 1 - t^2, \quad dt = d\sin x = \cos x dx.$

Thus we have

$$\int \frac{(\cos^3 x + \cos^5 x)dx}{\sin^2 x + \sin^4 x} = \int \frac{\cos^2 x(1 + \cos^2 x)\cos x dx}{\sin^2 x + \sin^4 x} = \int \frac{(1-t^2)(2-t^2)}{t^2 + t^4} dt.$$

Note that

$$\frac{(1-t^2)(2-t^2)}{t^2 + t^4} = \frac{2 - 3t^2 + t^4}{t^2 + t^4} = \frac{t^2 + t^4 + 2 - 4t^2}{t^2 + t^4} =$$
$$= 1 + \frac{2 - 4t^2}{t^2 + t^4} = 1 + \frac{2(1+t^2) - 6t^2}{t^2(1+t^2)} = 1 + \frac{2}{t^2} - \frac{6}{1+t^2}.$$

Therefore,

$$\int \frac{(1-t^2)(2-t^2)}{t^2 + t^4} dt = \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2}\right) dt = t - \frac{2}{t} -$$
$$- 6\arctan t + C = \sin x - \frac{2}{\sin x} - 6\arctan(snx) + C.$$

II. Integrals of the form $\int \sin^m x \cdot \cos^n x dx$

Here we will consider two cases which are of especial significance.

Case 1. At least one of the exponent *m* and *n* is an odd positive number. If *n* is an odd positive number, then we use the substitution $\sin x = t;$ now if *m* is odd positive number, the substitution $\cos x = t$.

Example. Find the integral $\int \sin^8 x \cdot \cos^7 x dx$.

Solution. Putting $\sin x = t$, $\cos x = dt,$ we obtain.

$$\int \sin^8 x \cdot \cos^7 x dx = \int \sin^8 x \cdot \cos^6 x \cdot \cos x dx = \int \sin^8 x (1-\sin^2 x)^3 d\sin x =$$
$$= \int t^8 (1-t^2)^3 dt = \int t^8 (1-3t^2+3t^4-t^6) dt = \int (t^8 - 3t^{10} + 3t^{12} - t^{14}) dt =$$
$$= \frac{t^9}{9} - 3\cdot\frac{t^{11}}{11} + 3\cdot\frac{t^{13}}{13} - \frac{t^{15}}{15} + C = \frac{1}{9}\sin^9 x - \frac{3}{11}\sin^{11} x + \frac{3}{13}\sin^{13} x - \frac{1}{15}\sin^{15} x + C.$$

Example. Find the integral $\int \sin^5 x \cdot \cos^6 x dx$.

Solution. Setting $\cos x = t,\ -\sin x dx = dt$, we get

$$\int \sin^5 x \cdot \cos^6 x dx = \int \sin^4 x \cdot \cos^6 x \cdot \sin x dx = \int (1-\cos^2 x)^2 \cos^6 x \sin x dx = -\int (1-t^2)^2 \cdot t^6 dt =$$
$$= -\int (1-2t^2+t^4) t^6 dt = -\int (t^6 - 2t^8 + t^{10}) dt = -\frac{t^7}{7} + 2\cdot\frac{t^9}{9} - \frac{t^{11}}{11} + C =$$
$$= -\frac{1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x + C.$$

Case 2. Both exponents *m* and *n* are even positive numbers. In this case the integrand must be transformed by mean of the formulas

$$\sin x \cdot \cos x = \frac{1}{2}\sin 2x;$$

$$\sin^2 x \cdot \cos^2 x = \frac{1}{4}\sin^2 2x;$$

$$\sin^2 x = \frac{1-\cos 2x}{2}; \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

Example. Find the integral $\int \sin^2 x \cdot \cos^2 x dx$.

Solution. We have

$$\int \sin^2 x \cdot \cos^2 x dx = \int \frac{1}{4}\sin^2 2x dx = \frac{1}{4}\int \frac{1-\cos 4x}{2} dx =$$
$$= \frac{1}{8}\int (1-\cos 4x) dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

Example. Find the integral $\int \sin^4 x \cdot \cos^4 x dx$.

Solution. We have

$$\int \sin^4 x \cdot \cos^4 x dx = \int (\sin^2 x \cdot \cos^2 x)^2 dx = \int \left(\frac{1}{4}\sin^2 2x\right)^2 dx = \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx =$$
$$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x) dx = \frac{1}{64}\int dx - \frac{1}{32}\int \cos 4x dx + \frac{1}{64}\int \frac{1+\cos 8x}{2} dx =$$
$$= \frac{1}{64}x - \frac{1}{128}\sin 4x + \frac{1}{128}x + \frac{1}{1024}\sin 8x + C = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$$

Example. Find the integral $\int \sin^2 x \cdot \cos^4 x dx$.

Solution. We have

$$\int \sin^2 x \cdot \cos^4 x dx = \int (\sin^2 x \cdot \cos^2 x)\cos^2 x dx = \int \left(\frac{1}{4}\sin^2 2x\right)\frac{1+\cos 2x}{2} dx =$$
$$\frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cdot \cos 2x dx = \frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{8}\int \sin^2 2x \cdot \frac{1}{2} d(\sin 2x) =$$
$$= \frac{1}{16}\int dx - \frac{1}{16}\int \cos 4x dx + \frac{1}{16}\int \sin^2 2x d(\sin 2x) = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

III. Integrals of the form $\int \tan^m x dx$ and $\int \cot^m x dx$

These integrals can be found with the aid the formula

$$\tan^2 x = \frac{1}{\cos^2 x} - 1; \quad \text{or } \cot^2 x = \frac{1}{\sin^2 x} - 1,$$

by means of which the degree of the tangent or cotangent is consecutively lowered

Example. Find the integral $\int \tan^5 x dx$.

Solution. We have

$$\int \tan^5 x dx = \int \tan^3 x \left(\frac{1}{\cos^2 x} - 1 \right) dx = \int \tan^3 x d \tan x - \int \tan^3 x dx =$$

$$= \frac{\tan^4 x}{4} - \int \tan x \left(\frac{1}{\cos^2 x} - 1 \right) dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C.$$

Example. Find the integral $\int \cot^8 x dx$.

Solution. We have

$$\int \cot^8 x dx = \int \cot^6 x \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\int \cot^6 x d\cot x - \int \cot^6 x dx =$$

$$= -\frac{\cot^7 x}{7} - \int \cot^4 x \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\frac{\cot^7 x}{7} + \int \cot^4 x d\cot x + \int \cot^4 x dx =$$

$$= -\frac{\cot^7 x}{7} + -\frac{\cot^5 x}{5} + \int \cot^2 x \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\frac{\cot^7 x}{7} + -\frac{\cot^5 x}{5} -$$

$$-\int \cot^2 x d\cot x - \int \cot^2 dx == -\frac{\cot^7 x}{7} + \frac{\cot^5 x}{5} - \frac{\cot^3 x}{3} - \int \left(\frac{1}{\sin^2 x} - 1 \right) dx =$$

$$= -\frac{\cot^7 x}{7} + \frac{\cot^5 x}{5} - \frac{\cot^3 x}{3} + \cot x + x + C$$

IV. Integrals of the form $\int \sin mx \cos nx dx,\ \int \cos mx \cos nx dx,\ \int \sin mx \sin nx dx$

These integrals can be found with the aid of the following trigonometric formulas

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right],$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right],$$

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

by means of which make it possible to represent a product of trigonometric functions as their sum.

Example. Find the integral $\int \sin 3x \cdot \cos 7x dx$.

Solution. We have

$$\int \sin 3x \cdot \cos 7x dx = \frac{1}{2}\int [\sin 10x + \sin(-4x)]dx = \frac{1}{2}\int \sin 10x dx -$$

$$-\frac{1}{2}\int \sin 4x dx = -\frac{1}{20}\cos 10x + \frac{1}{8}\cos 4x + C$$

Example. Find the integral $\int \cos x \cdot \cos 3x \cdot \cos 5x dx$.

Solution. We have

Lecture 13
Definite Integral

There are various problems of geometry and physics leading to the notion of the definite integral: determining the area of a plane figure, computing the work of a variable force, finding the distance travelled by a body with a given velocity, and many other problems.

Definition. Suppose a function $f(x)$ is defined on a closed interval [*a*,*b*]. Let us divide the interval [*a*,*b*] into *n* arbitrary parts by points $a = x_0 < x_1 < x_2 < ... < x_n = b$, choose an arbitrary point ξ_k in each subinterval $[x_{k-1}, x_k]$ and find the length of every such subinterval: $\Delta x_k = x_k - x_{k-1}$.

The **integral sum** for the function $f(x)$ on the closed interval $[a,b]$ is the sum of the form

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + ... + f(\xi_n)\Delta x_n$$

Each term (summand) of the integral sum is equal to the area of the rectangle with the base Δx_k and the altitude $f(\xi_k)$ and which is obviously equal to $f(\xi_k)\Delta x_k$ (Fig4.1).

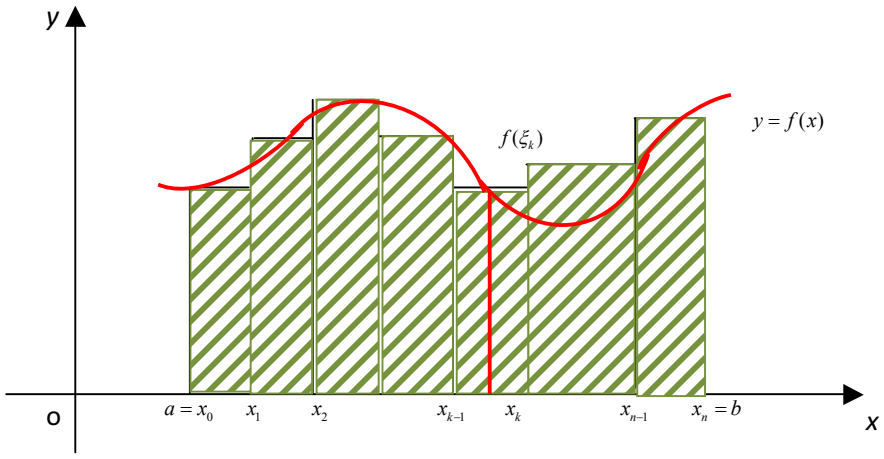


Fig 4.1

The integral sum is the area of the entire step figure consisted of these rectangles.

The definite integral of the function $f(x)$ on the closed interval $[a,b]$ is the limit of the integral sum under the condition that the length of the greatest subinterval tends to zero:

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

The numbers *a* and *b*, are called respectively, **the lower limit** and **the upper limit** of integration. As in the case of the indefinite integral, the function $f(x)$ is called the **integrand,** the expression $f(x)dx$ is called the **element of integration,** the

interval $[a,b]$ is called the **interval of integration**, and the variable *x* is the **variable of integration.**

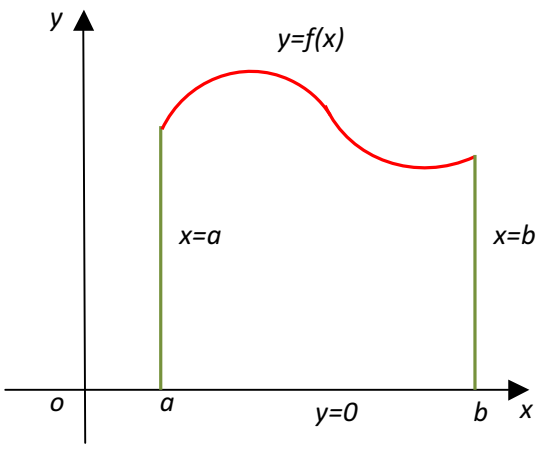
Definition. A function $f(x)$ for which the definite integral exists is called **integrable.**

The process of forming the definite integral shows that the symbol $\int_a^b f(x)dx$ is a certain **number.** Its value only depends on the integrand $f(x)$ and on the numbers *a* and *b* determining the interval of integration.

Geometrical interpretation of the definite integral

Let us suppose that $f(x)>0$ on the interval $[a,b]$.

Definition. A figure bounded by the graph of a function $y=f(x)$, by the straight lines $y=0$, $x=a$ and $x=b$ is called **a curvilinear trapezoid.**



If $f(x)>0$ on $[a,b]$, then, in geometrical interpretation, the definite integral $\int_a^b f(x)dx$ is the area of a curvilinear trapezoid (Fig.4.2):

Fig.4.2

$$A = \int_a^b f(x)dx$$

Basic Properties of a Definite Integral

1. A definite integral does not depend on the notation of the variable of integration, i.e.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \ldots$$

2. $\int_a^b dx = b - a$.

It is clear that any integral sum for the function $f(x) \equiv 1$ is equal to $b - a$:

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = \sum_{k=1}^{n} \Delta x_k = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n =$$
$$= x_1 - x_0 + x_2 - x_1 + \ldots + x_n - x_{n-1} = x_n - x_0 = b - a$$

3. If the upper and the lower limits of the definite integral are interchanged the integral only changer its sign:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

4. If the limits of integration coincide then the integral is equal to zero, i.e

$$\int_a^a f(x)dx = 0$$

5.For any a,b,c there holds the formula

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (*)$$

Proof. Let $a < c < b$.

Since the limit of the integral sum does not depend on the way the interval $[a,b]$ is divided into parts, we can divide the interval so that the point *C* is always a point of division. Then the integral sum can be represented in the form

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k}{}' f(\xi_k)\Delta x_k + \sum_{k=1}^{n}{}'' f(\xi_k)\Delta x_k$$

where the first sum on the right-hand side includes all the terms corresponding to the points of division of the interval $[a,c]$ and the second sum includes all the terms corresponding to the points of division of the interval $[c,b]$. The first and the second sums are, respectively, integral sums for the function $f(x)$ corresponding to the intervals $[a,c]$ and $[c,b]$.

If the number of the points of division increases indefinitely and the length of the greatest subinterval tends to zero for the whole interval $[a,b]$ the same obviously takes place for the intervals $[a,c]$ and $[c,b]$; therefore, passing to the limit as max $\Delta x_k \to 0$, we get

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{\max \Delta x_k \to 0} \sum_{k}{}' f(\xi_k)\Delta x_k + \lim_{\max \Delta x_k \to 0} \sum_{k}{}'' f(\xi_k)\Delta x_k$$

The first limit on the right-hand side is equal to the integral from *a* to *c* and the second to the integral from *c* to *b*, and thus the required equality is obtained.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Equality (*) also holds in the case when the point c lies outside the interval $[a,b]$ to the right of it, i.e. $a < b < c$, or to the left of it, i.e. $c < a < b$.

Indeed, let $a < b < c$. As has been proved, since *b* lies between *a* and *c* we have

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

whence

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx,$$

Now, interchanging the limits of integration in the second integral on the right-hand side we obtain

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

which is what we had to prove.

It can similarly be proved that equality (*) is valid for $c < a < b$.

6. The integral of a sum of a finite number of functions is equal to the sum of the integrals of these functions:

$$\int_a^b \left[f(x) + g(x)\right]dx = \int_a^b f(x)dx + \int_{ac}^b g(x)dx,$$

$$\int_a^b [f_1(x) + \ldots + f_n(x)]dx = \int_a^b f_1(x)dx + \ldots + \int_{ac}^b f_n(x)dx.$$

Proof. According to the definition of the integral, we have

$$\int_a^b [f(x) + g(x)]dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^n [f(\xi_k) + g(\xi_k)]\Delta x_k = \lim_{\max \Delta x_k \to 0} \left[\sum_{k=1}^n f(\xi_k)\Delta x_k + \sum_{k=1}^n g(\xi_k)\Delta x_k \right] =$$

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^n f(\xi_k)\Delta x_k + \lim_{\max \Delta x_k \to 0} \sum_{k=1}^n g(\xi_k)\Delta x_k = \int_a^b f(x)dx + \int_{ac}^b g(x)dx$$

7. A constant factor in the integrand can be taken outside the integral sign:

$$\int_a^b Cf(x)dx = C\int_a^b f(x)dx$$

where *C* is constant.

Proof. According to the definition of the integral, we have

$$\int_a^b Cf(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^n Cf(\xi_k)\Delta x_k = C \lim_{\max \Delta x_k \to 0} \sum_{k=1}^n f(\xi_k)\Delta x_k = C\int_a^b f(x)dx.$$

8. If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx \geq 0.$

Proof. Let $f(x) \geq 0$ on $[a,b]$. Then in the integral sum all the summands are nonnegative, and hence

$$\sum_{k=1}^n f(\xi_k)\Delta x_k \geq 0.$$

It follows that

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k \geq 0$$

since the limit of a nonnegative quantity cannot be negative.

9. If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Proof. We have

$$f(x) \geq g(x) \Rightarrow f(x) - g(x) \geq 0 \Rightarrow \int_a^b [f(x) - g(x)]dx \geq 0 \Rightarrow$$

$$\Rightarrow \int_a^b f(x)dx - \int_a^b g(x)dx \geq 0 \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx \ .$$

10. If *m* and *M* are, respectively, the least and the greatest values of the function $f(x)$ in the interval $[a,b]$ ($a < b$) then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \ .$$

Proof. We have $m \leq f(x) \leq M$ for any $x \in [a,b]$.

According to the previous property we can integrate the inequality and hence

$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$$

$$m\int_a^b dx \leq \int_a^b f(x)dx \leq M\int_a^b dx$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M \cdot (b-a) .$$

The geometrical meaning of the inequalities we have just proved is the following:

The area of a curvilinear trapezoid is greater than the area of the rectangle with altitude equal to *m* and is less than the area of the rectangle with altitude equal to *M* (Fig.4.3)

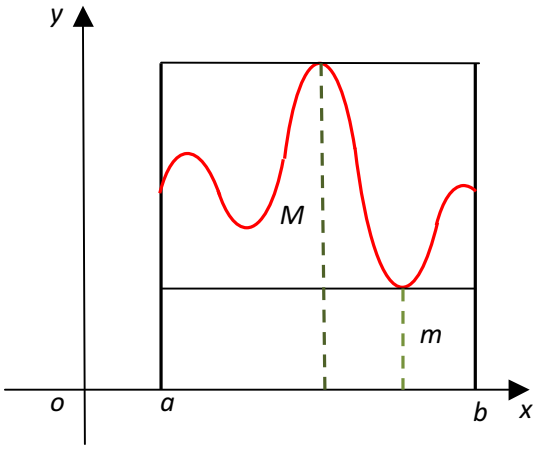


Fig.4.3

11. If at every point $x \in [a,b]$ the inequalities

$$\varphi(x) \le f(x) \le \psi(x)$$

are fulfilled then

$$\int_a^b \varphi(x)dx \le \int_a^b f(x)dx \le \int_a^b \psi(x)dx\,.$$

This means that an inequality between functions implies an inequality of the same sense between their definite integrals, or, briefly speaking, that it is allowable to **integrate inequalities.**

Estimate of a Definite Integral

There holds the following theorem:

Theorem. The absolute value of the definite integral does not exceed the integral of the absolute value of the integrand function

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Proof. For any *x* we have

$$-|f(x)| \le f(x) \le |f(x)|$$

It follows that

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

that is

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Mean Value Theorem

The definite integral possesses the following important property:

Theorem. Let $f(x)$ be a continuous function on $[a,b]$.

Then there exists at least one point $x = \xi \in (a,b)$ such that

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Proof. If the function $f(x)$ is constant, i.e. $f(x) = C$ the formula is obvious

$$\int_a^b Cdx = C\int_a^b dx = C(b-a)$$

Let us suppose $f(x)$ is nonconstant, then according to the property 10 of the definite integral we have

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$

and, hence,

$$\frac{1}{b-a}\int_a^b f(x)dx = \mu$$

where μ is a number lying between the least value *m* and the greatest value *M* of the function $f(x)$ on $[a,b]$.

Since $f(x)$ is continuous on $[a,b]$ according to well-known property of continuous function the function $f(x)$ assumes all the intermediate values between *m* and *M*. Therefore, there exist at least one point $\xi \in (a,b)$ at which $f(\xi) = \mu$

i.e.

$$\frac{1}{b-a}\int_a^b f(x)dx = f(\xi)$$

In geometrical interpretation the mean value theorem shows that the definite integral of a continuous function is equal to the area of some rectangle with altitude $f(\xi)$ (Fig.4.4).

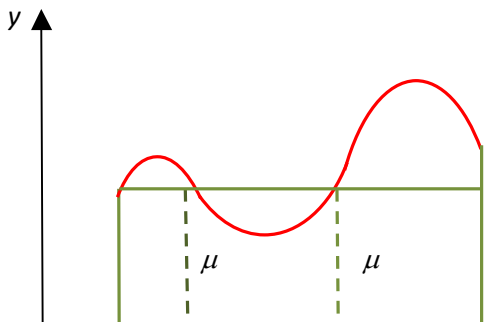


Fig.4.4

Lecture 14

Integral with Variable Upper Limit

Let us consider the function of the form

$$F(x) = \int_a^x f(t)dt$$

which is called the integral with variable upper limit. Here the lower limit is fixed (constant), and the upper limit *x* is variable.

For our further aims it is important to study some properties of this function $F(x)$.

Continuity of the Integral with Variable Upper Limit

Theorem. If $f(x)$ is integrable on $[a,b]$, then the function $F(x)$ is continuous on $[a,b]$.

Proof. Let us take an arbitrary $x \in [a,b]$ and give the argument *x* an increment Δx.

Let us consider the difference $F(x+\Delta x) - F(x)$. We have

$$F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt =$$
$$= \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt.$$

Now estimating the expression $|F(x+\Delta x) - F(x)|$ we get

$$|F(x+\Delta x) - F(x)| = \left| \int_x^{x+\Delta x} f(t)dt \right| \le \int_x^{x+\Delta x} |f(t)|dt \le \left| \int_x^{x+\Delta x} |f(t)|dt \right|$$

Since the function $f(x)$ is integrable on $[a,b]$ then the function is bounded on $[a,b]$, that is

$|f(t) \le C|$ for any $t \in [a,b]$, *C*=constant

Taking it into account, we obtain

$$|F(x+\Delta x) - F(x)| \le \left| \int_x^{x+\Delta x} C dt \right| = C \left| \int_x^{x+\Delta x} dt \right| = C \cdot |\Delta x|.$$

It follows that $|F(x+\Delta x) - F(x)| \to 0$ as $\Delta x \to 0$, and, hence $\lim_{\Delta x \to 0} F(x+\Delta x) = F(x)$ which means that the function $F(x)$ is continuous at *x*.

Differentiability of the Integral with Variable Upper Limit

Theorem. If a function *f(x)* is continuous on $[a,b]$ then the derivative of the integral with respect to its upper limit is equal to the integral:

$$F'(x)=\left(\int_a^x f(t)dt\right)'=f(x).$$

In other words: an integral with variable upper limit is an antiderivative of its integrand.

Proof. We have to prove that $\lim\limits_{\Delta x\to 0}\dfrac{\Delta F}{\Delta x}=f(x)$.

For the further it should be noted that

$$\int_x^{x+\Delta x} dt=\Delta x\Rightarrow \frac{1}{\Delta x}\int_x^{x+\Delta x} dt=1\Rightarrow f(x)=\frac{1}{\Delta x}\int_x^{x+\Delta x} f(x)dt.$$

We have

$$\frac{\Delta F}{\Delta x}=\frac{F(x+\Delta x)-F(x)}{\Delta x}=\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt.$$

Let us consider the difference $\dfrac{\Delta F}{\Delta x}-f(x)$.

$$\frac{\Delta F}{\Delta x}-f(x)=\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt-f(x)=\frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)dt\right]-\frac{1}{\Delta x}\int_x^{x+\Delta x} f(x)dt=\frac{1}{\Delta x}\int_x^{x+\Delta x}\left[f(t)-f(x)\right]dt$$

Since the function $f(x)$ is continuous, it is clear that $\Delta x\to 0\Rightarrow t\to x\Rightarrow f(t)\to f(x)\Rightarrow f(t)-f(x)\to 0$. It follows that for any $\varepsilon>0$ $\left|f(t)-f(x)\right|<\varepsilon$ and hence

$$\left|\frac{\Delta F}{\Delta x}-f(x)\right|=\left|\frac{1}{\Delta x}\int_{x}^{x+\Delta x}[f(t)-f(x)]dt\right|\le\frac{1}{|\Delta x|}\cdot\left|\int_{x}^{x+\Delta x}|f(t)-f(x)|dt\right|<\frac{1}{|\Delta x|}\cdot\varepsilon\cdot|\Delta x|=\varepsilon$$

or $\left|\frac{F}{\Delta x}-f(x)\right|<\varepsilon$

which means that $\lim_{\Delta x\to 0}\frac{F}{\Delta x}=f(x),$ i.e. $F'(x)=f(x)$

Theorem. Any function continuous on [*a*,*b*] possesses an antiderivative.

The Newton-Leibniz Theorem

Theorem. The definite integral is equal to the difference of the values of any antiderivative of the integrand corresponding to the upper and lower of integral.

$\int_{a}^{b}f(x)dx=\Phi(b)-\Phi(a),$ where $\Phi'(x)=f(x).$

In other words:

The definite integral is equal to the increment of any antiderivative of the integrand gained on the interval of integration.

This result is one of the most important theorem in mathematics. It is called the Newton-Leibniz theorem, and the equality

$$\int_{a}^{b}f(x)dx=\Phi(b)-\Phi(a)$$

is referred to as the **Newton-Leibniz formula.**

Proof. Let us consider the function

$$F(x) = \int_a^x f(t)dt$$

and let $\Phi(x)$ be arbitrary antiderivative of the function $f(x)$. Since any two antiderivatives differ by a constant we get

$$F(x) = \Phi(x) + C$$

or

$$\int_a^x f(t)dt = \Phi(x) + C$$

Here putting $x = a$ we obtain

$\int_a^x f(t)dt = \Phi(x) + C$ or $0 = \Phi(a) + C, \qquad C = -\Phi(a)$

and hence

$$\int_a^x f(t)dt = \Phi(x) - \Phi(a).$$

Now setting $x = b$ we finally get

$$\int_a^x f(t)dt = \Phi(b) - \Phi(a)$$

which is what we had to prove.

Methods of Evaluating Definite Integral

1.**Newton-Leibniz formula:**

$$\int_a^x f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

2. **Integration by parts:**

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu,$$

where $u = u(x)$ are continuously differentiable functions on the interval [*a*,*b*].

Proof. We have

$$d(uv) = vdu + udv$$

Integrating both sides of this equality from *a* to *b* we obtain

$$\int_a^b d(uv) = \int_a^b vdu + \int_a^b udv$$

whence

$$\int_a^b udv = \int_a^b d(uv) - \int_a^b vdu$$

or

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu.$$

3. **Change of variable in the definite integral:**

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\cdot\varphi'(t)dt \tag{1}$$

where $x=\varphi(t)$ and $\varphi'(t)$ are continuous on interval $[\alpha,\beta]$, $\varphi(\alpha)=a,\ \varphi(\beta)=b,$

and $f[\varphi(t)]$ is a function continuous on $[\alpha,\beta]$.

Proof. Let us suppose $F(x)$ is an antiderivative of $f(x)$. Then

$$\int f(x)dx = F(x)+C, \tag{2}$$

$$\int f[\varphi(t)]\cdot\varphi'(t)dt = F[\varphi(t)]+C. \tag{3}$$

From (2) we get:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

From (3) we obtain:

$$\int_\alpha^\beta f[\varphi(t)]\cdot\varphi'(t)dt = F\big[\varphi(t)\big]_\alpha^\beta = F[\varphi(\beta)-\varphi(\alpha)] = F(b)-F(a)$$

Comparing the equalities we get the formula (1).

4. If $f(x)$ is an odd function, i.e. $f(-x)=-f(x)$, then

$$\int_{-a}^{a} f(x)dx = 0.$$

Proof. We have

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$

Now, changing the variable of integration in the first integral on the right-hand side by means of the substitution $x = -t$ we get

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)dx = 0.$$

5. If $f(x)$ is an even function, i.e. $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Proof. We do the same here

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)dx = 2 \cdot \int_{0}^{a} f(x)dx.$$

Let us consider several examples.

1.Evaluate the integral $\int_{-1}^{2} x^2 dx$.

Solution. We have

$$\int_{-1}^{2} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{2} = \frac{2^3}{3} - \frac{(-1)^3}{3} = \frac{8}{3} + \frac{1}{3} = 3$$

2. Evaluate the integral $\int_0^{\frac{\pi}{2}} \cos x dx$.

Solution. We have

$$\int_0^{\frac{\pi}{2}} \cos x dx = \sin x\Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

3. Evaluate the integral $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}$.

Solution. We have

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = \tan x\Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 0.$$

4. Calculate the integral $\int_0^1 xe^{-x} dx$.

Solution. We use here the method of integration by parts.

We put $u = x,\ dv = e^{-x}dx$, whence $du = dx,\ v = -e^{-x}$.

Then

$$\int_0^1 xe^{-x}dx = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx = -e^{-1} - e^{-x}\Big|_0^1 = -2e^{-1} + 1 = \frac{e-2}{e}.$$

5. Calculate the integral $\int_0^{\frac{\pi}{2}} e^x \cos x dx$.

Solution. We use here method of integration by parts.

We set $u = e^x, dv = \cos x dx$, whence $du = e^x dx,\ \ v = \sin x$. Then

$$\int_0^{\frac{\pi}{2}} e^x \cos x dx = e^x \cdot \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x dx = e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x dx$$

Applying here the same method, we put $e^x = u,\ \sin x dx = dv$, whence $du = e^x dx,\ \ v = -\cos x$. Then we obtain

$$\int_0^{\frac{\pi}{2}} e^x \cos x dx = e^{\frac{\pi}{2}} + e^x \cdot \cos\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x dx = e^{\frac{\pi}{2}} - 1 - \int_0^{\frac{\pi}{2}} e^x \cos x dx.$$

It follows that

$$\int_0^{\frac{\pi}{2}} e^x \cos x dx = \frac{e^{\frac{\pi}{2}} - 1}{2}.$$

6. Calculate the integral $\int_1^e \frac{\ln^2 x}{x} dx$.

Solution. We use here the method for change of variable.

We put $\ln x = t,\ \ $ then $\frac{dx}{x} = dt\ \ .$

It is clear that if $x = 1,\ \ $ then $t = 0\ $; if $x = e,\ \ $ then $t = 1\ \ .$

Consequently,

$$\int_0^e \frac{\ln^2 x}{x} dx = \int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

7. Evaluate the integral $\int_3^8 \frac{x dx}{\sqrt{1+x}}$.

Solution. We set $\sqrt{1+x}=t$;then $x=t^2-1,\ dx=2tdt$.

If $x=3$, then $t=2$; if $x=8$,then $t=3$. Therefore,

$$\int_3^8 \frac{xdx}{\sqrt{1+x}}=\int_2^3 \frac{(t^2-1)\cdot 2tdt}{t}=2\int_2^3 (t^2-1)dt=2\cdot\frac{t^3}{3}\Big|_2^3-2t\Big|_2^3=$$
$$=\frac{2}{3}(27-8)-2(3-2)=\frac{2}{3}\cdot 19-2=\frac{32}{3}=10\frac{2}{3}$$

8. Calculate the integral $\int_{-\pi}^{\pi}\sin^3 x\cos^2 dx$.

Solution. Here the integrand is an odd function.

$$\int_{-\pi}^{\pi}\sin^3 x\cos^2 dx=0$$

Lecture 15

Improper Integrals

Up to now, when speaking of definite integrals we assumed that the interval of integration was finite and closed and that the integrand was continuous.

Improper integrals are:

(1) integrals with infinite limits of integration;

(2) integrals of unbounded functions.

Improper Integrals with Infinite Limits

We introduce the following definition

Definition. The improper integral $\int_a^{+\infty} f(x)dx$ of the function $f(x)$ over the interval $[a;+\infty)$ is the limit of the integral $\int_a^b f(x)dx$ as $b \to +\infty$ provided that this limit exists:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx$$

If the limit exists and is finite, the improper integral is called **convergent;** now if the limit does not exist or is equal to infinity, it is called **divergent.**

Similarly,

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

and

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)dx$$

Examples.

1.Compute the improper integral $\int_0^{+\infty} e^{-x}dx$.

Solution. We have

$$\int_0^{+\infty} e^{-x}dx = \lim_{b\to+\infty} \int_0^b e^{-x}dx = \lim_{b\to+\infty}(-e^{-x})\Big|_0^b =$$

$$= \lim_{b\to+\infty}(-e^{-b} + e^0) = \lim_{b\to+\infty}\left(1 - \frac{1}{e^b}\right) = 1$$

Consequently, the improper integral $\int_0^{+\infty} e^{-x}dx$ is equal to 1 and hence, it is convergent.

2. Compute the improper integral $\int\limits_{-\infty}^{-1}\frac{dx}{x^2}$.

Solution. We find

$$\int\limits_{-\infty}^{-1}\frac{dx}{x^2}=\lim_{a\to-\infty}\int\limits_{a}^{-1}\frac{dx}{x^2}=\lim_{a\to-\infty}\left(-\frac{1}{x}\right)\Bigg|_{a}^{-1}=\lim_{a\to-\infty}\left(1+\frac{1}{a}\right)=1$$

that is, the improper integral converges.

3. Compute the improper integral $\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}$.

Solution. The integrand is even, therefore

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}=2\int\limits_{0}^{+\infty}\frac{dx}{1+x^2}.$$

Then we have

$$\int\limits_{0}^{+\infty}\frac{dx}{1+x^2}=\lim_{b\to+\infty}\int\limits_{0}^{b}\frac{dx}{1+x^2}=\lim_{b\to+\infty}\arctan x\Big|_{0}^{b}=$$

$$=\lim_{b\to+\infty}(\arctan b-\arctan 0)=\lim_{b\to+\infty}\arctan b=\frac{\pi}{2}$$

Thus, finally we get $\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}=\pi$,

That is, the improper integral is convergent.

4. Compute the improper integral $\int\limits_{0}^{+\infty}\cos x dx$ (or establish its divergence).

Solution. We have

$$\lim_{b\to+\infty}\int\limits_{0}^{b}\cos x dx=\lim_{b\to+\infty}\sin x\Big|_{0}^{b}=\lim_{b\to+\infty}(\sin b-\sin 0)=\lim_{b\to+\infty}\sin b.$$

The limit $\lim_{b\to+\infty}\sin b$ does not exist since $\sin b$ oscillates between -1 and 1 as $b\to+\infty$. Consequently, the improper integral is divergent.

Comparison Tests for Convergence or Divergence of Improper Integrals with Infinite Limits

To investigate the convergence of improper integrals, we use one of the comparison tests. To this end we will prove the following theorems (comparison tests).

Theorem 1. Let the inequalities

$$0 \le f(x) \le \varphi(x)$$

hold for all the values of $x \ge a$.

Then, if the integral $\int_a^{+\infty}\varphi(x)dx$ is convergent the integral $\int_a^{+\infty}f(x)dx$ is convergent as well.

Proof. Let us suppose that the integral $\int_a^{+\infty}\varphi(x)dx$ is convergent and is equal to *M*. It is clear that

$$\int_a^b \varphi(x)dx \le \int_a^{+\infty}\varphi(x)dx = M \text{ for any } b$$

and, hence

$$\int_a^b \varphi(x)dx \le M \text{ for any } b.$$

According to the property on integrating inequalities, we get

$$f(x) \le \varphi(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b \varphi(x)dx \le M.$$

It follows that $I(b) = \int_a^b f(x)dx$ is a bounded function, and therefore the integral $\int_a^{+\infty} f(x)dx$ is convergent. The theorem has been proved.

Theorem 2. Let the inequalities

$$0 \le f(x) \le \varphi(x)$$

hold for all the values of $x \ge a$.

Then, if the integral $\int_a^{+\infty} f(x)dx$ is divergent the integral $\int_a^{+\infty} \varphi(x)dx$ is also divergent.

Proof. Now suppose that the integral $\int_a^{+\infty} f(x)dx$ is divergent.

Then the increasing function $\int_a^b f(x)dx$ tends to $+\infty$ as $b \to +\infty$. But since

$$\int_a^b \varphi(x)dx \ge \int_a^b f(x)dx,$$

the function $\int_a^b \varphi(x)dx$ also tends to $+\infty$ as $b \to +\infty$ and hence, the integral $\int_a^{+\infty} \varphi(x)dx$ is divergent, which is what we had to prove.

Example 1. Test the integral $\int_a^{+\infty} \frac{dx}{x^p}$ for convergence.

Solution. If p=1 we have by definition

$$\int_a^{+\infty} \frac{dx}{x} = \lim_{A \to +\infty} \int_a^A \frac{dx}{x} = \lim_{A \to +\infty} \ln|x| \Big|_a^A = \lim_{A \to +\infty} (\ln A - \ln|a|) = +\infty$$

since $\ln A \to +\infty$ as $A \to +\infty$. Consequently, at *p*=1 the integral $\int_a^{+\infty} \frac{dx}{x}$ is divergent.

If $p \neq 1$we have by definition

$$\int_a^{+\infty} \frac{dx}{x^p} = \lim_{A\to+\infty} \int_a^A \frac{dx}{x^p} = \lim_{A\to+\infty} \int_a^A x^{-p} dx = \lim_{A\to+\infty} \left[\frac{1}{-p+1} x^{-p+1} \right]_a^A =$$

$$= \lim_{A\to+\infty} \left[\frac{1}{-p+1} A^{-p+1} - \frac{1}{-p+1} a^{-p+1} \right] =$$

$$= \frac{1}{-p+1} \lim_{A\to+\infty} \frac{1}{A^{p-1}} + \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}$$

Assume that $p > 1$, then $p-1>0$and $\lim_{A\to+\infty} \frac{1}{A^{p-1}} = 0$.

It follows that at $p > 1$the integral converges and

$$\int_a^{+\infty} \frac{dx}{x^p} = \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}.$$

Assume now that $p < 1$, then $p-1<0$and $\lim_{A\to+\infty} \frac{1}{A^{p-1}} = +\infty$. This means that at $p < 1$ the integral diverges.

Therefore, finally we get

$$\int_a^{+\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} \cdot \frac{1}{a^{p-1}} & at \quad p > 1 \\ +\infty & at \quad p \leq 1 \end{cases}$$

Thus, we have

$$\int_a^{+\infty} \frac{dx}{x^p} = \begin{cases} \text{converges} & at \quad p > 1 \\ \text{diverges} & at \quad p \leq 1. \end{cases}$$

Example 2. Test the integral $\int_1^{+\infty} \frac{dx}{1+x^8}$ for convergence.

Solution. The integrand

$$f(x)=\frac{dx}{1+x^8}$$

is smaller over the interval of integration than $\varphi(x)=\frac{1}{x^8}$ since

$$\frac{1}{1+x^8}\le\frac{1}{x^8},$$

and the integral $\int_1^{+\infty}\frac{dx}{x^8}$ is convergent (see the previous example).

Therefore, using the comparison test (theorem 1), it follows that, the given integral converges too.

Example 3. Test the integral $\int_1^{+\infty}\frac{dx}{\sqrt{1+x}\cdot\sqrt[3]{1+x^2}}$ for convergence.

Solution. It is clear that the inequality

$$\frac{dx}{\sqrt{1+x}\cdot\sqrt[3]{1+x^2}}<\frac{1}{x^{\frac{1}{2}}\cdot x^{\frac{2}{3}}}=\frac{1}{x^{\frac{7}{6}}}$$

holds for all $x\ge 1,$ and the integral $\int_1^{+\infty}\frac{dx}{x^{\frac{7}{6}}}$ is convergent since $p=\frac{7}{6}>1$. Consequently, applying the comparison test (theorem 1), we get that the given integral is also convergent.

Example 4. Test the integral $\int_1^{+\infty}\frac{\sqrt{x}}{1+x}dx$ for convergence.

Solution. It is obvious that the inequality

$$\frac{\sqrt{x}}{1+x}>\frac{\sqrt{x}}{x+x}=\frac{1}{2\sqrt{x}}$$

holds for all x>1 and the integral $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ is divergent because $p = \frac{1}{2} < 1$. Therefore, according to the comparison test (theorem 2), the given integral is also divergent.

Theorem 3. If the integral $\int_a^{+\infty} |f(x)|dx$ of the absolute value of the function $f(x)$ is convergent the integral $\int_a^{+\infty} f(x)dx$ is also convergent.

In this case the integral $\int_a^{+\infty} f(x)dx$ is said to be absolutely convergent. Therefore, theorem 3 means that the absolutely convergent integral is also convergent.

Example 5. The integrals $\int_0^{+\infty} \frac{\cos x}{1+x^2}dx$ and $\int_0^{+\infty} \frac{\sin x}{1+x^2}dx$ are absolutely convergent because the absolute values of the integrands do not exceed the positive function $\frac{1}{1+x^2}$ and the integral $\int_0^{+\infty} \frac{dx}{1+x^2}$ is convergent. Indeed, we have

$$\left|\frac{\cos x}{1+x^2}\right| = \frac{|\cos x|}{1+x^2} < \frac{1}{1+x^2}$$

and, similarly,

$$\left|\frac{\sin x}{1+x^2}\right| \le \frac{1}{1+x^2}$$

By definition

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_a^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} \arctan x\Big|_0^b = \lim_{b\to+\infty} \arctan b = \frac{\pi}{2}$$

and hence, the integral is convergent.

Now, applying the comparison test (theorem 1), we get that the integrals

$$\int_0^{+\infty} \frac{|\cos x|}{1+x^2} dx \quad \text{and} \quad \int_0^{+\infty} \frac{|\sin x|}{1+x^2} dx$$

are convergent, and it means that the integrals

$$\int_0^{+\infty} \frac{\cos x}{1+x^2} dx \quad \text{and} \quad \int_0^{+\infty} \frac{\sin x}{1+x^2} dx$$

are absolutely convergent.

It is easy to prove the following theorem.

Theorem 4. If the integral $\int_{-\infty}^{+\infty} |f(x)| dx$ is convergent, the integrals $\int_{-\infty}^{+\infty} f(x)\cos x dx$ and $\int_{-\infty}^{+\infty} f(x)\sin x dx$ are absolutely convergent.

Proof. Indeed, it clear that the absolute values of the integrands obviously do not exceed $|f(x)|$

$$|f(x)\cos x| = |f(x)| \cdot |\cos x| \le |f(x)|$$

$$|f(x)\sin x| = |f(x)| \cdot |\sin x| \le |f(x)|.$$

Therefore, according to the comparison test (theorem 1) we get that both given integrals are absolutely convergent, which is what we had to prove.

It should be noted, the fact that the integral $\int_a^{+\infty} |f(x)| dx$ is divergent does not enable us to judge upon the convergence of the integral $\int_a^{+\infty} f(x) dx$ because it can be either divergent or convergent.

Definition. The improper integral $\int_a^{+\infty} f(x)dx$ is said to be **conditionally (not absolutely) convergent** if the integral $\int_a^{+\infty} f(x)dx$ is convergent and the integral $\int_a^{+\infty} |f(x)|dx$ is divergent.

Example 6. Let us consider the Dirichlet integral $\int_0^{+\infty} \frac{\sin x}{x}dx$. It can be shown that the Dirichlet integral is conditionally convergent because this integral is convergent and its numerical value is found by means of some special techniques:

$$\int_0^{+\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

while the integral of the absolute value of the integrand is divergent:

$$\int_0^{+\infty} \left|\frac{\sin x}{x}\right|dx = +\infty.$$

Thus, the Dirichlet integral is conditionally divergent.

Improper Integrals of Unbounded Functions

Let $y = f(x)$ be a continuous function for all $x \in [a,b)$ (i.e., $a \le x < b$) having an infinite discontinuity at $x = b$. It is clear that the ordinary definition of the definite integral is inapplicable here because the function $f(x)$ is unbounded in [*a*,*b*]. Let us first take the ordinary definite integral

$$I(\varepsilon) = \int_a^{b-\varepsilon} f(x)dx \quad \text{where } \varepsilon > 0$$

and then make ε tend to zero: $\lim_{\varepsilon \to 0} I(\varepsilon)$. Then $I(\varepsilon)$ either tends to a finite limit or has no finite limit (in the latter case it either tends to infinity or has no limit at all).

We introduce the following definition

Definition. The improper integral $\int_a^b f(x)dx$ of a function $f(x)$ continuous for $a \le x < b$ and unbounded as $x \to b$ is the limit of the integral $\int_a^{b-\varepsilon} f(x)dx$ as $\varepsilon \to 0 \ \ (\varepsilon > 0)$ provided this limit exists:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)dx, \qquad \varepsilon > 0$$

If the limit exists we say that the improper integral is convergent. If this limit does not exist the equality becomes meaningless, and the improper integral written on the left is said to be divergent.

Similary, if the function $f(x)$ has an infinite discontinuity only at $x = a$ of the interval [*a*,*b*] and is continuous for all $x \in (a,b]$ (i.e., $a < x \le b$) then, we put by definition

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^b f(x)dx, \text{where } \delta > 0.$$

provided this limit exists.

Definition. If the function $f(x)$ possesses an infinite discontinuity at a point $c \in (a,b)$ and is continuous at $a \le x < c$ and $c < x \le b$, then, by definition

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0} \int_{c+\delta}^b f(x)dx.$$

The improper integral $\int_a^b f(x)dx$ (where $f(c) = \infty,\ a < c < b$) is said to be **convergent,** if both limits exist on the right-hand side of the equality, and **divergent,** if either of them does not exist.

Now let us consider some examples.

Example 7. Compute the improper integral $\int_0^1 \frac{dx}{x}$.

Solution. The integrand $f(x)=\frac{1}{x}$ at the point $x=0$ is unbounded and, therefore, we have

$$\int_0^1 \frac{dx}{x}=\lim_{\delta\to 0}\int_0^1 \frac{dx}{x}=\lim_{\delta\to 0}\ln|x|\Big|_\delta^1=\lim_{\delta\to 0}(\ln 1-\ln\delta)=+\infty,$$

that is, the improper integral diverges.

Examples. Calculate the following improper integrals:

1. $\int_0^a \frac{dx}{\sqrt{x}}=\lim_{\delta\to 0}\int_\delta^a \frac{dx}{\sqrt{x}}=\lim_{\delta\to 0}2\sqrt{x}\Big|_\delta^a=\lim_{\delta\to 0}(2\sqrt{a}-2\sqrt{\delta})=2\sqrt{a}$.

2. $\int_0^1 \frac{dx}{1-x}=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{dx}{1-x}=\lim_{\varepsilon\to 0}\left[-\ln(1-x)\right]_0^{1-\varepsilon}=$
$=\lim_{\varepsilon\to 0}\left[-\ln\varepsilon+\ln 1\right]=-\lim_{\varepsilon\to 0}\ln\varepsilon=+\infty$

That is, the integral is divergent

3. $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$. In this example the integrand has infinite discontinuities at both end points of the interval of integration. We have

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}=\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}\int_{-1+\delta}^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}}=\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}\arcsin x\Big|_{-1+\delta}^{1-\varepsilon}=$$
$$=\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}\left[\arcsin(1-\varepsilon)-\arcsin(-1+\delta)\right]=$$
$$=\lim_{\varepsilon\to 0}\arcsin(1-\varepsilon)-\lim_{\delta\to 0}\arcsin(-1+\delta)=$$
$$=\arcsin 1-\arcsin(-1)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.$$